Series y residuos

Credit: This notes are 100% from chapter 6 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

Cauchy's integral formula for derivatives indicates that if a function f is analytic at a point z_0 , then it possesses derivatives of all orders at that point. As a consequence of this result we shall see that f can always be expanded in a power series centered at that point. On the other hand, if f fails to be analytic at z_0, we may still be able to expand it in a different kind of series known as a Laurent series. The notion of Laurent series leads to the concept of a residue, and this, in turn, leads to yet another way of evaluating complex and, in some instances, real integrals.

Sequences and Series

Sequences A sequence $\{z_n\}$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers C. If $\lim_{n\to\infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent**. In other words, $\{z_n\}$ converges to the number L if for each positive real number ε an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$. A sequence that is not convergent is said to be **divergent**.

Example: The sequence $\{\frac{i^{n+1}}{n}\}$ is convergent, $\lim_{n\to\infty} \frac{i^{n+1}}{n} = 0$.

Theorem (6.1): Criterion for Convergence A sequence $\{z_n\}$ converges to a complex number $L = a + ib$ if and only if $\Re(z_n)$ converges to $\Re(L) = a$ and $\Im(z_n)$ converges to $\Im(L) = b$.

Example Consider the sequence $\{\frac{3+in}{n+i2n}\}$
Solution: it converges since,

$$\begin{aligned} z_n &= \frac{3+in}{n+i2n} = \frac{2n^2+3n}{5n^2} + i\frac{n^2-6n}{5n^2} && (1)\\ \Re(z_n) &\to \frac{2}{5} && (2)\\ \Im(z_n) &\to \frac{1}{5} && (3)\end{aligned}$$

as $n \to \infty$.

Series An infinite series or series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_n + \cdots \tag{4}$$

is convergent if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + \cdots + z_n \tag{5}$$

converges. If $S_n \to L$ as $n \to \infty$, we say that the series converges to L or that the sum of the series is L.

Geometric Series A geometric series is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \tag{6}$$

the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1} \tag{7}$$

When an infinite series is a geometric series, it is always possible to find a formula for S_n:

$$zS_n = az + az^2 + \cdots + az^n \tag{8}$$

$$S_n - zS_n = (az + az^2 + \cdots + az^n) - (a + az + az^2 + \cdots + az^{n-1}) \tag{9}$$

$$S_n(1-z) = a - az^n = a(1 - z^n) \Rightarrow S_n = a\frac{1-z^n}{1-z} \tag{10}$$

for $n \to \infty$, $z^n \to 0$ whenever $|z| < 1$, then $S_n \to a/(1-z)$, then

$$a + az + az^2 + \cdots + az^{n-1} + \cdots = \frac{a}{1-z} \tag{11}$$

A geometric series diverges when $|z| \geq 1$.

Special Geometric Series .
(i) For $a = 1$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{12}$$

(ii) For $a = 1$ and $z \to -z$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \tag{13}$$

(iii) For $a = 1$

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} \tag{14}$$

(iv) By writing $\frac{1-z^n}{1-z} = \frac{1}{1-z} + \frac{-z^n}{1-z}$ we get

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^{n-1} + \frac{z^n}{1-z} \tag{15}$$

Example The infinite series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \cdots \tag{16}$$

is a geometric series with $a = \frac{1}{5}(1+2i)$ and $z = \frac{1}{5}(1+2i)$. Since $|z| = \sqrt{5}/5 < 1$, the series is convergent

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+i2}{5}}{1 - \frac{1+i2}{5}} = i\frac{1}{2} \tag{17}$$

Theorem (6.2): A Necessary Condition for Convergence If $\sum_{k=1}^{\infty} z_k$ converges, then $lim_{n\to\infty} z_n = 0$.

Theorem (6.3): The nth Term Test for Divergence If $lim_{n\to\infty} z_n \neq 0$, then $\sum_{k=1}^{\infty} z_k$ diverges.

Absolute and Conditional Convergence An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **absolutely convergent** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be **conditionally convergent** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

Absolute convergence implies convergence.

Example The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent since the series $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right|$ is the same as the real convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Tests for Convergence Two of the most frequently used tests for convergence of infinite series are given in the next theorems.

Theorem (6.4): Ratio Test Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L \tag{18}$$

(i) If $L < 1$, then the series converges absolutely.
(ii) If $L > 1$ or $L = \infty$, then the series diverges.
(iii) If $L = 1$, the test is inconclusive.

Theorem (6.5): Root Test Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L \tag{19}$$

(i) If $L < 1$, then the series converges absolutely.
(ii) If $L > 1$ or $L = \infty$, then the series diverges.
(iii) If $L = 1$, the test is inconclusive.

Power Series The notion of a power series is important in the study of analytic functions. An infinite series of the form

$$\sum_{k=1}^{\infty} a_k(z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \tag{20}$$

where the coefficients a_k are complex constants, is called a power series in $z - z_0$. It is said to be centered at z_0, called the center of the series.

Circle of Convergence Every complex power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ has a **radius of convergence** and a **circle of convergence**, which is the circle centered at z_0 of largest radius $R > 0$ for which the series converges at every point within the circle $|z-z_0| = R$. A power series converges absolutely at all points z within its circle of convergence, that is, for all z satisfying $|z-z_0| < R$, and diverges at all points z exterior to the circle, that is, for all z satisfying $|z-z_0| > R$. The radius of convergence can be:
(i) $R = 0$ (in which case the serie converges only at its center $z = z_0$),
(ii) R a finite positive number (in which case the serie converges at all interior points of the circle $|z-z_0| = R$), or
(iii) $R = \infty$ (in which case the serie converges for all z).
A power series may converge at some, all, or at none of the points on the actual circle of convergence.

Example Consider the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test,

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1}|z| = |z| \tag{21}$$

Thus the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$. On the circle of convergence $|z| = 1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the well-known divergent harmonic series.

For a power series $\sum_{k=1}^{\infty} a_k(z-z_0)^k$, the limit depends only on the coefficients a_k. Thus, if
(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, the radius of convergence is $R = 1/L$. Esto es así porque para que la serie converja el cociente $\lim \frac{a_{n+1}|z-z_0|^{n+1}}{a_n|z-z_0|^n} = \lim \frac{a_{n+1}}{a_n} R = LR < 1$.
(ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, the radius of convergence is $R = \infty$
(iii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the radius of convergence is $R = 0$

Similar conclusions can be made for the root test.

Example Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}(z-1-i)^k$. With the identification $a_n = (-1)^{n+1}/n!$ we have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 \tag{22}$$

Hence the radius of convergence is ∞; the power series with center $z_0 = 1+i$ converges absolutely for all z, that is, for $|z-1-i| < \infty$.

Example Consider the power series $\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5}\right)^k (z-2i)^k$. With $a_n = \left(\frac{6n+1}{2n+5}\right)^n$, the root test gives

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{6n+1}{2n+5} = 3 \tag{23}$$

Then, the radius of convergence of the series is $R = 1/3$. The circle of convergence is $|z-2i| = 1/3$; the power series converges absolutely for $|z-2i| < 1/3$.

The Arithmetic of Power Series Some facts

- A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be multiplied by a nonzero complex constant c without affecting its convergence or divergence.

- A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely within its circle of convergence. As a consequence, within the circle of convergence the terms of the series can be rearranged and the rearranged series has the same sum L as the original series.
- A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ and $\sum_{k=0}^{\infty} b_k(z-z_0)^k$ can be added and subtracted by adding or subtracting like terms. In symbols:

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \pm \sum_{k=0}^{\infty} b_k(z-z_0)^k = \sum_{k=0}^{\infty}(a_k \pm b_k)(z-z_0)^k \tag{24}$$

  If both series have the same nonzero radius R of convergence, the radius of convergence of $\sum_{k=0}^{\infty}(a_k \pm b_k)(z-z_0)^k$ is R.
- Two power series can (with care) be multiplied and divided.

Taylor Series

Throughout the discussion in this section we will assume that a power series has either a positive or an infinite radius R of convergence.

Differentiation and Integration of Power Series The three theorems that follow indicate a function f that is defined by a power series is continuous, differentiable, and integrable within its circle of convergence.

Theorem (6.6): Continuity A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ represents a continuous function f within its circle of convergence $|z-z_0| = R$.

Theorem (6.7): Term-by-Term Differentiation A power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be differentiated term by term within its circle of convergence $|z-z_0| = R$.

It follows as a corollary to Theorem 6.7 that a power series defines an infinitely differentiable function within its circle of convergence and each differentiated series has the same radius of convergence R as the original power series.

Theorem (6.8): Term-by-Term Integration A power series$\sum_{k=0}^{\infty} a_k(z-z_0)^k$ can be integrated term-by-term within its circle of convergence $|z-z_0| = R$, for every contour C lying entirely within the circle of convergence.

The theorem states that

$$\int_C \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int_C (z-z_0)^k dz \tag{25}$$

whenever C lies in the interior of $|z-z_0| = R$. Indefinite integration can also be carried out term by term:

$$\int \sum_{k=0}^{\infty} a_k(z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \int (z-z_0)^k dz = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1} + constant \tag{26}$$

The ratio test can be used to be prove that both

$$\sum_{k=0}^{\infty} a_k(z-z_0)^k \tag{27}$$

and

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1}(z-z_0)^{k+1} \tag{28}$$

have the same circle of convergence $|z - z_0| = R$.

Taylor Series Suppose a power series represents a function f within $|z - z_0| = R$, that is,

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots \tag{29}$$

It follows from Theorem 6.7 that the derivatives of f are the series

$$\begin{aligned} f'(z) &= \sum_{k=1}^{\infty} a_k k (z-z_0)^{k-1} = a_1 + 2a_2(z-z_0) + 3a_3(z-z_0)^2 + \cdots \\ f''(z) &= \sum_{k=2}^{\infty} a_k k(k-1)(z-z_0)^{k-2} = 2 \cdot 1 a_2 + 3 \cdot 2 a_3(z-z_0) + \cdots \\ &\vdots \end{aligned}$$

Since the power series $f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$ represents a differentiable function f within its circle of convergence $|z - z_0| = R$, where R is either a positive number or infinity, we conclude that a *power series represents an analytic function* within its circle of convergence. By evaluating the derivatives at $z = z_0$ we get,

$$f(z_0) = a_0 \tag{30}$$

$$f'(z_0) = 1!a_1 \tag{31}$$

$$f''(z_0) = 2!a_2 \tag{32}$$

$$f'''(z_0) = 3!a_3 \tag{33}$$

$$\vdots \tag{34}$$

In general,

$$a_n = \frac{f^{(n)}(z_0)}{n!} \tag{35}$$

with $n \geq 0$. Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \tag{36}$$

This series is called the **Taylor series** for f centered at z_0.

Maclaurin series It is the Taylor series for $z_0 = 0$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \tag{37}$$

Theorem (6.9): Taylor's Theorem Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k \tag{38}$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.
Proof: See pag. 316 of the book.

We can find the radius of convergence of a Taylor series as the distance from the center z_0 of the series to the nearest isolated singularity of f. Where, an isolated singularity is a point at which f fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point. For example, $z = 5i$ is an isolated singularity of $f(z) = 1/(z - 5i)$. If the function f is entire, then the radius of convergence $R = \infty$.

Some Important Maclaurin Series

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{39}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-)^k \frac{z^{2k+1}}{(2k+1)!} \tag{40}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-)^k \frac{z^{2k}}{(2k)!} \tag{41}$$

Example Suppose the function $f(z) = (3 - i)/(1 - i + z)$ is expanded in a Taylor series with center $z_0 = 4 - 2i$. What is its radius of convergence R ?
Solution: Observe that the function is analytic at every point except at $z = -1 + i$, which is an isolated singularity of f. The distance from $z = -1 + i$ to $z_0 = 4 - 2i$ is

$$|z - z_0| = \sqrt{(-1-4)^2 + (1-(-2))^2} = \sqrt{34} = R \tag{42}$$

The power series expansion of a function, with center z_0, is unique. On a practical level this means that a power series expansion of an analytic function f centered at z_0 , irrespective of the method used to obtain it, is the Taylor series expansion of the function.

Example For example, we can obtain

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-)^k \frac{z^{2k}}{(2k)!} \tag{43}$$

by simply differentiating

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-)^k \frac{z^{2k+1}}{(2k+1)!} \tag{44}$$

term by term.

Example For example, the Maclaurin series for e^{z^2} can be obtained by replacing the symbol z in

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{45}$$

by z^2,i.e.

$$e^{z^2} = 1 + \frac{(z^2)}{1!} + \frac{(z^2)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(z^2)^k}{k!} \tag{46}$$

$$= 1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^{2k}}{k!} \tag{47}$$

Example Find the Maclaurin expansion of $f(z) = \frac{1}{(1-z)^2}$.
Solution: From,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \tag{48}$$

valid for $|z| < 1$, we differentiate both sides with respect to z to get

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1} \tag{49}$$

Since we are using Theorem 6.7, the radius of convergence of the last power series is the same as the original series, $R = 1$.

Example We can often build on results such as the above one. For example, if we want the Maclaurin expansion of $f(z) = \frac{z^3}{(1-z)^2}$, we simply multiply the above equation by z^3:

$$\frac{z^3}{(1-z)^2} = z^3 + 2z^4 + 3z^5 + \cdots = \sum_{k=1}^{\infty} kz^{k+2} \tag{50}$$

Its radius of convergence is still $R = 1$.

Example Expand $f(z) = 1/(1-z)$ in a Taylor series with center $z_0 = 2i$.
Solution: By using the geometric series we have

$$\frac{1}{1-z} = \frac{1}{1-z+2i-2i} = \frac{1}{1-2i-(z-2i)} \tag{51}$$

$$= \frac{1}{1-2i} \frac{1}{1-\frac{z-2i}{1-2i}} \tag{52}$$

Next, we use the power series by replacing $z \to \frac{z-2i}{1-2i}$,

$$\frac{1}{1-z} = \frac{1}{1-2i} \frac{1}{1-\frac{z-2i}{1-2i}} \tag{53}$$

$$= \frac{1}{1-2i}\left[1 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^2 + \frac{z-2i}{1-2i} + \left(\frac{z-2i}{1-2i}\right)^3 + \cdots\right] \tag{54}$$

$$= \frac{1}{1-2i} + \frac{z-2i}{(1-2i)^2} + \frac{(z-2i)^2}{(1-2i)^3} + \cdots \tag{55}$$

Because the distance from the center $z_0 = 2i$ to the nearest singularity $z = 1$ is $\sqrt{5}$, we conclude that the circle of convergence is $|z - 2i| = \sqrt{5}$.

Remark As a consequence of Theorem 5.11, we know that an analytic function f is infinitely differentiable. As a consequence of Theorem 6.9, we know that an analytic function f can always be expanded in a power series with a nonzero radius R of convergence. In real analysis, a function f can be infinitely differentiable, but it may be impossible to represent it by a power series.

Laurent Series

If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or **singular point** of the function. For example, the complex numbers $z = \pm 2i$ are singularities of the function $f(z) = z/(z^2+4)$; the nonpositive x-axis and the branch point $z = 0$ are singular points of Lnz. In this section we will be concerned with a new kind of "power series" expansion of f about an **isolated singularity** z_0.

Isolated Singularities Suppose that $z = z_0$ is a singularity of a complex function f. The point $z = z_0$ is said to be an isolated singularity of the function f if there exists some deleted neighborhood, or punctured open disk, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic. For example, $z = \pm 2i$ are isolated singularities of $f(z) = z/(z^2+4)$. On the other hand, the branch point $z = 0$ is not an isolated singularity of Lnz. We say that a singular point $z = z_0$ of a function f is **nonisolated** if every neighborhood of z_0 contains at least one singularity of f other than z_0. For example, the branch point $z = 0$ is a nonisolated singularity of Lnz.

Series with negative powers If $z = z_0$ is a singularity of a function f, then certainly f cannot be expanded in a power series with z_0 as its center. However, about an isolated singularity $z = z_0$, it is possible to represent f by a series involving both negative and nonnegative integer powers of $z - z_0$,

$$\begin{aligned} f(z) &= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 \cdots \\ &= \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k \end{aligned} \tag{56}$$

The series with negative powers is called **principal part** and will converge for $\left|\frac{1}{z-z_0}\right| < r^*$, i.e. $|z - z_0| > 1/r^* = r$. The part of nonnegative powers is called the **analytic part** and will converge for $|z - z_0| < R$. Then, the sum converges when z is a point in an annular domains defined by $r < |z - z_0| < R$.

Example The function $f(z) = \sin z/z^4$ is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series. Since $\sin z$ is an entire function with Maclaurin series given by,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \tag{57}$$

for $|z| < \infty$, we have

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots \tag{58}$$

The analytic part of this series converges for $|z| < \infty$. The principal part is valid for $|z| > 0$. Thus, it converges for all z except at $z = 0$, i.e. $0 < |z| < \infty$.

Theorem (6.10): Laurent's Theorem Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \tag{59}$$

valid for $r < |z - z_0| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds \tag{60}$$

with $k = 0, \pm 1, \cdots$, where C is a simple closed curve that lies entirely within D and has z_0 in its interior. See Figure 1

Proof: See pag. 327 in the book.

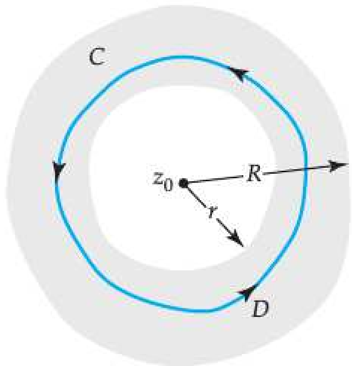

Figure 1: (From the book)

Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; that is, the series we obtain is unique.

In the case when $a_{-k} = 0$ for $k = 1, 2, 3, \cdots$, the principal part of the Laurent series is zero and it reduces to a Taylor series. Thus, a Laurent expansion can be considered as a generalization of a Taylor series.

Example Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains:
(a) $0 < |z| < 1$
(b) $1 < |z|$
(c) $0 < |z - 1| < 1$
(d) $1 < |z - 1|$

(a) We expand the geometric series $1/(1 - z)$ valid for $|z| < 1$,

$$f(z) = -\frac{1}{z}\frac{1}{1-z} \tag{61}$$

$$= -\frac{1}{z}[1 + z + z^2 + z^3 + \cdots] \tag{62}$$

$$= -\frac{1}{z} - 1 - z - z^2 - \cdots \tag{63}$$

which converges for $0 < |z| < 1$.

(b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1 \Rightarrow 1 < |z|$,

$$\begin{aligned}
f(z) &= \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} && (64)\\
&= \frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right] && (65)\\
&= \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots && (66)
\end{aligned}$$

(c) We rewrite $f(z)$ and proceed like in (a),

$$\begin{aligned}
f(z) &= \frac{1}{(1-1+z)(z-1)} && (67)\\
&= \frac{1}{z-1}\frac{1}{1+(z-1)} && (68)\\
&= \frac{1}{z-1}[1-(z-1)+(z-1)^2-(z-1)^3+\cdots] && (69)\\
&= \frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots && (70)
\end{aligned}$$

The requirement that $z \neq 1$ is equivalent to $0 < |z-1|$, and the geometric series in brackets converges for $|z-1| < 1$. Thus the last series converges for z satisfying $0 < |z-1|$ and $|z-1| < 1$, that is, for $0 < |z-1| < 1$.

(d) Proceeding as in part (b), we write

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)(1+(z-1))} && (71)\\
&= \frac{1}{(z-1)^2}\frac{1}{1+\frac{1}{z-1}} && (72)\\
&= \frac{1}{(z-1)^2}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right] && (73)\\
&= \frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\cdots && (74)
\end{aligned}$$

Because the series within the brackets converges for $|1/(z-1)| < 1$, the final series converges for $1 < |z-1|$.

Example Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for:
(a) $0 < |z-1| < 2$ and
(b) $0 < |z-3| < 2$.
Solution:

(a) We need to express $z-3$ in terms of $z-1$

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)^2(z-3)} && (75)\\
&= \frac{1}{(z-1)^2}\left[\frac{1}{-2+(z-1)}\right] && (76)\\
&= \frac{-1}{2(z-1)^2}\left[\frac{1}{1-\frac{z-1}{2}}\right] && (77)\\
&= \frac{-1}{2(z-1)^2}\left[1+\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\left(\frac{z-1}{2}\right)^3+\cdots\right] && (78)\\
&= -\frac{1}{2(z-1)^2}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots && (79)
\end{aligned}$$

(b) To obtain powers of $z-3$, we write $z-1=2+(z-3)$ and

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)^2(z-3)} && (80)\\
&= \frac{1}{z-3}[2+(z-3)]^{-2} && (81)\\
&= \frac{1}{4(z-3)}\left[1+\frac{z-3}{2}\right]^{-2} && (82)\\
&= \frac{1}{4(z-3)}\left[1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2+\cdots\right] && \\
&= \frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{1}{8}(z-3)^2+\cdots && (83)
\end{aligned}$$

where we have used the binomial series ($(1+z)^\alpha = 1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots$ valid for $|z|<1$). The binomial expansion is valid for $|(z-3)/2|<1$, i.e. $|z-3|<2$.

Example Expand $f(z)=\frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0<|z|<1$.
Solution: by partial fractions we write

$$\begin{aligned}
f(z) &= \frac{8z+1}{z(1-z)} && (84)\\
&= \frac{1}{z}+\frac{9}{1-z} && (85)\\
&= \frac{1}{z}+9+9z+9z^2+\cdots && (86)
\end{aligned}$$

the geometric series converges for $|z|<1$, but due to the term $1/z$, the resulting Laurent series is valid for $0<|z|<1$.

Example Expand $f(z)=e^{3/z}$ in a Laurent series valid for $0<|z|<\infty$.
Solution: For all finite $|z|<\infty$, is valid the expansion

$$e^z = 1+z+\frac{z^2}{2!}+\cdots \qquad (87)$$

We obtain the Laurent series f by simply replacing $z\to 3/z$, for $z\neq 0$,

$$e^{3/z} = 1+\frac{3}{z}+\frac{3^2}{2!z^2}+\cdots \qquad (88)$$

valid for $0<|z|<\infty$.

Zeros and Poles

We will assign different names to the isolated singularity $z = z_0$ according to the number of terms in the principal part of the Laurent series.

Classification of Isolated Singular Points. An isolated singular point $z = z_0$ of a complex function f is given a classification depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms (Table 1):
(i) **Removable singularity:** If the principal part is zero, that is, all the coefficients a_{-k} are zero, then $z = z_0$ is called a removable singularity.
(ii) **Pole:** If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a pole. If, in this case, the last nonzero coefficient is a_{-n} , $n \geq 1$, then we say that $z = z_0$ is a pole of order n. If $z = z_0$ is pole of order 1, then the principal part contains exactly one term with coefficient a_{-1}. A pole of order 1 is commonly called a **simple pole**.
(iii) **Essential singularity:** If the principal part contains an infinitely many nonzero terms, then $z = z_0$ is called an essential singularity.

$z = z_0$	**Laurent series for** $0 < \lvert z - z_0 \rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\frac{a_{-1}}{z-z_0} + a_0 + a_1(z - z_0) + \cdots$
Essential singularity	$\cdots + \frac{a_2}{(z-z_0)^{-2}} + a_0 + a_1(z - z_0) + \cdots$

Table 1: From the book.

Removable Singularity In the series,

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{89}$$

that all the coefficients in the principal part of the Laurent series are zero. Hence $z = 0$ is a removable singularity of the function $f(z) = (\sin z)/z$.

If a function f has a removable singularity at the point $z = z_0$, then we can always supply an appropriate definition for the value of $f(z_0)$ so that f becomes analytic at $z = z_0$. For instance, since the right-hand side of the series expansion of $(\sin z)/z$ is 1 when we set $z = 0$, it makes sense to define $f(0) = 1$. Hence the function $f(z) = (sinz)/z$, as given by

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \tag{90}$$

is now defined and continuous at every complex number z. Indeed, f is also analytic at $z = 0$ because it is represented by the Taylor series $1 - z^2/3! + z^4/5! - \cdots$ centered at 0.

Example .
(i)

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots \quad (91)$$

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots \quad (92)$$

for $0 < |z| < \infty$, the $z = 0$ is a simple pole of the function $f(z) = (\sin z)/z^2$ and a pole of order 3 of the function $g(z) = (\sin z)/z^4$.
(ii) The expansion of f of the Example 3 of Section 6.3 valid for $0 < |z-1| < 2$ was given by the equation

$$f(z) = \frac{1}{(z-1)^2(z-3)} = -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{z-1}{16} - \cdots \quad (93)$$

Then, $z = 1$ is a pole of order 2.
(iii) The value $z = 0$ is an essential singularity of $f(z) = e^{3/z}$.

Zeros A number z_0 is zero of a function f if $f(z_0) = 0$. We say that an analytic function f has a **zero of order** n or a **zero of multiplicity** n at $z = z_0$ if

$$f(z_0) = 0 \quad (94)$$

$$f'(z_0) = 0 \quad (95)$$

$$f''(z_0) = 0 \quad (96)$$

$$\vdots \quad (97)$$

$$f^{(n-1)}(z_0) = 0 \quad (98)$$

but

$$f^{(n)}(z_0) \neq 0 \quad (99)$$

A zero of order 1 is called a **simple zero**.

Example For $f(z) = (z-5)^3$ we see that $f(5) = 0$, $f'(5) = 0$, $f''(5) = 0$, but $f'''(5) = 6 \neq 0$. Thus f has a zero of order (or multiplicity) 3 at $z_0 = 5$.

Theorem (6.11): Zero of Order n A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^n \phi(z) \quad (100)$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$

Example The analytic function $f(z) = z \sin z^2$ has a zero at $z = 0$. If we replace z by z^2 in the series expansion of $\sin z$ we get

$$f(z) = z \sin^2 z = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \cdots \quad (101)$$

$$= z^3[1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots] \quad (102)$$

$$= z^3 \phi(z) \quad (103)$$

with $\phi(0) = 1 \neq 0$, then $z = 0$ is a zero of f of order 3.

Theorem (6.12): Pole of Order n A function f analytic in a punctured disk $0 < |z-z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written

$$f(z) = \frac{\phi(z)}{(z-z_0)^n} \tag{104}$$

where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

More about zeros A zero $z = z_0$ of an analytic function f is isolated in the sense that there exists some neighborhood of z_0 for which $f(z) \neq 0$ at every point z in that neighborhood except at $z = z_0$. As a consequence, if z_0 is a zero of a nontrivial analytic function f , then the function $1/f(z)$ has an isolated singularity at the point $z = z_0$.

Theorem (6.13): Pole of Order n If the functions g and h are analytic at $z = z_0$ and h has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $f(z) = g(z)/h(z)$ has a pole of order n at $z = z_0$.

Examples .
(i) The rational function

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4} \tag{105}$$

shows that the denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at any of these points, it follows that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.
(ii) The value $z = 0$ is a zero of order 3 of $z \sin z^2$. Then, we conclude that the reciprocal function $f(z) = 1/(z \sin z^2)$ has a pole of order 3 at $z = 0$.

Remarks .
(i) From the preceding discussion, it should be intuitively clear that if a function f has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ from any direction and we can write $\lim_{z\to z_0} f(z) = \infty$.
(ii) A function f is **meromorphic** if it is analytic throughout a domain D, except possibly for poles in D. It can be proved that a meromorphic function can have at most a finite number of poles in D. For example, the rational function $f(z) = 1/(z^2 + 1)$ is meromorphic in the complex plane.

Residues and Residue Theorem

Residue The coefficient a_1 of $1/(z - z_0)$ in the Laurent series is called the residue of the function f at the isolated singularity z_0, noted as $a_1 = Res(f(z), z_0)$.

Examples .
(i) $z = 1$ is a pole of order two of the function $f(z) = \frac{1}{(z-1)^2(z-3)}$. From the Laurent series obtained above valid for the deleted neighborhood of $z = 1$ defined by $0 < |z - 1| < 2$,

$$f(z) = \frac{-1/2}{(z-1)^2} + \frac{-1/4}{z-1} - \frac{1}{8} - \frac{z-1}{16} - \cdots ... \tag{106}$$

we have $Res(f(z), 1) = -1/4$.
(ii) $z = 0$ is an essential singularity of $f(z) = e^{3/z}$. From its Laurent series

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \cdots \tag{107}$$

valid for $0 < |z| < \infty$, we get $Res(f(z), 0) = 3$.

The following theorem gives a way to obtain the residues of a function f without the necessity of expanding f in a Laurent series.

Theorem (6.14): Residue at a Simple Pole If f has a simple pole at $z = z_0$, then

$$Res(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{108}$$

Prof: Since f has a simple pole at $z = z_0$, its Laurent expansion convergent on a punctured disk $0 < |z - z_0| < R$ has the form

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots \tag{109}$$

where $a_1 \neq 0$. By multiplying both sides of this series by $z - z_0$ and then taking the limit as $z \to z_0$ we obtain the above relation.

Theorem (6.15): Residue at a Pole of Order n If f has a pole of order n at $z = z_0$, then

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{110}$$

Proof: See pag. 344 of the book.

Example The function $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$. Find the residues.
Solution: Since $z = 3$ is a simple pole, we have:

$$Res(f(z), 3) = \lim_{z \to 3} (z - 3) f(z) = \lim_{z \to 3} \frac{1}{(z-1)^2} = \frac{1}{4} \tag{111}$$

For the pole of order 2, we have

$$\begin{aligned} Res(f(z), 1) &= \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}} (z-1)^2 f(z) && (112) \\ &= \lim_{z \to 1} \frac{d}{dz} (z-1)^2 \frac{1}{(z-1)^2(z-3)} && (113) \\ &= \lim_{z \to 1} \frac{d}{dz} \frac{1}{(z-3)} && (114) \\ &= \lim_{z \to 1} \frac{-1}{(z-3)^2} = \frac{-1}{4} && (115) \end{aligned}$$

When f is not a rational function, calculating residues by means of the above limits can sometimes be tedious. An alternative residue formula can be obtain if the function f can be written as a quotient $f(z) = g(z)/h(z)$, where g and h are analytic at $z = z_0$. If $g(z_0) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and

$$Res(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \tag{116}$$

Proof: Let us write the derivative of h

$$h'(z) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{h(z)}{z - z_0} \tag{117}$$

by using the definition of residues and $h(z_0) = 0$, we get

$$\begin{aligned} Res(f(z), z_0) &= \lim_{z \to z_0} (z - z_0) f(z) && (118) \\ &= \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} && (119) \\ &= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z)}{z - z_0}} && (120) \\ &= \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - h(z_0)}{z - z_0}} && (121) \\ &= \frac{g(z_0)}{h'(z_0)} && (122) \end{aligned}$$

Example The polynomial $z^4 + 1$ can be factored as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where $z_1 = e^{i\pi/4}$, $z_2 = e^{3i\pi/4}$, $z_3 = e^{5i\pi/4}$, and $z_4 = e^{7i\pi/4}$ are its four distinct roots. Then, the function $f(z) = 1/(z^4 + 1)$ has four simple poles. By using $Res(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$ we get

$$\begin{aligned} Res(f(z), z_1) &= \frac{1}{4z_1^3} = \frac{1}{4} e^{-3i\pi/4} = -\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}} && (123) \\ Res(f(z), z_2) &= \frac{1}{4z_2^3} = \frac{1}{4} e^{-9i\pi/4} = \frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}} && (124) \\ Res(f(z), z_3) &= \frac{1}{4z_3^3} = \frac{1}{4} e^{-15i\pi/4} = \frac{1}{4\sqrt{2}} + i\frac{1}{4\sqrt{2}} && (125) \\ Res(f(z), z_4) &= \frac{1}{4z_4^3} = \frac{1}{4} e^{-21i\pi/4} = -\frac{1}{4\sqrt{2}} + i\frac{1}{4\sqrt{2}} && (126) \end{aligned}$$

Alternatively we can use the expression $\lim_{z \to z_0} (z - z_0) f(z)$ for each, pole

$$Res(f(z), z_i) = \lim_{z \to z_i} (z - z_i) \frac{1}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \quad (127)$$

for example,

$$\begin{aligned} Res(f(z), z_1) &= \lim_{z \to z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \\ &= \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \\ &= \frac{1}{(e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})} \end{aligned}$$

and then, work out the above expression to reduce it to $-\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}$.

Theorem (6.16): Cauchy's Residue Theorem Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of isolated singular points $z_1, z_2, \cdots, z_n$ within C, then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k) \quad (128)$$

Proof: Suppose $C_1, C_2, \cdots, C_n$ are circles centered at $z_1, z_2, \cdots, z_n$, respectively. Suppose further that each circle C_k has a radius r_k small enough so that $C_1, C_2, \cdots, C_n$ are mutually

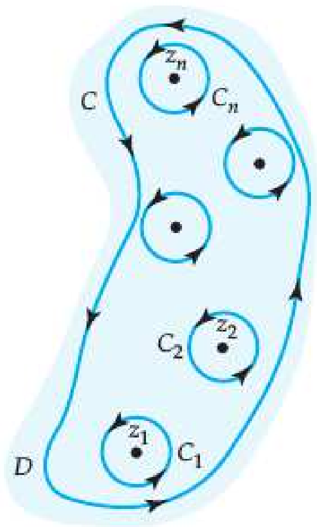

Figure 2: n singular points within contour C (From the book)

disjoint and are interior to the simple closed curve C, Fig. 2. We known from earlier develop that $\oint_{C_k} f(z)dz = 2\pi i Res(f(z), z_k)$, and so by Theorem 5.5 we have

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k) \tag{129}$$

Example Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$ for the following two contours:
(i) a rectangle defined by $x = 0$, $x = 4$, $y = -1$, $y = 1$,
(ii) the circle $|z| = 2$.
Solution:
(i) Since both $z = 1$ and $z = 3$ are poles within the rectangle we have

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i[Res(f(z), 1) + Res(f(z), 3)] \tag{130}$$

$$= 2\pi i \left[\frac{-1}{4} + \frac{1}{4}\right] = 0 \tag{131}$$

$$\tag{132}$$

(ii) Since only the pole $z = 1$ lies within the circle $|z| = 2$, we have

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i Res(f(z), 1) \tag{133}$$

$$= 2\pi i \left(\frac{-1}{4}\right) = -i\frac{\pi}{2} \tag{134}$$

Example Evaluate $\oint_C \frac{2z+6}{z^2+4}dz$ where the contour C is the circle $|z - i| = 2$:
Solution: By factoring the denominator as $z^2 + 4 = (z - 2i)(z + 2i)$ we see that the integrand has simple poles at $-2i$ and $2i$. Because only $2i$ lies within the contour C, we get

$$\oint_C \frac{2z+6}{z^2+4}dz = 2\pi i Res(f(z), 2i) \tag{135}$$

$$= 2\pi i \left(\frac{3+2i}{2i}\right) = \pi(3 + i2) \tag{136}$$

Example Evaluate $\oint_C \frac{e^z}{z^4+5z^3} dz$ where the contour C is the circle $|z| = 2$:
Solution: By factoring the denominator as $z^4 + 5z^3 = z^3(z + 5)$ we see that the integrand has a pole of order 3 at $z = 0$ and a single pole at $z = -5$. But only the first one is inside C, then

$$\begin{aligned}
\oint_C \frac{e^z}{z^4+5z^3} dz &= 2\pi i Res(f(z), 0) && (137)\\
&= 2\pi i \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 \frac{e^z}{z^4+5z^3} && (138)\\
&= \pi i \lim_{z\to 0} \frac{d^2}{dz^2} z^3 \frac{e^z}{z^3(z+5)} && (139)\\
&= \pi i \lim_{z\to 0} \frac{d^2}{dz^2} \frac{e^z}{(z+5)} && (140)\\
&= \pi i \lim_{z\to 0} \frac{(z^2+8z+17)e^z}{(z+5)^3} && (141)\\
&= i\frac{17\pi}{125} && (142)
\end{aligned}$$

Example Evaluate $\oint_c \tan z dz$, where the contour C is the circle $|z| = 2$.
Solution: The integrand $f(z) = \tan z = \sin z/\cos z$ has simple poles at the points where $\cos z = 0$, i.e. $z = (2n+1)\pi/2$, $n = 0, \pm 1, \cdots$. Since only $-\pi/2$ and $\pi/2$ are within the circle $|z| = 2$, we have

$$\oint_C \tan z dz = 2\pi i \left[Res(f(z), -\pi/2) + Res(f(z), \pi/2)\right] \qquad (143)$$

With the identifications $g(z) = \sin z$, $h(z) = \cos z$, and $h'(z) = -\sin z$, we get

$$\begin{aligned}
\oint_C \tan z dz &= 2\pi i \left[\frac{\sin(-\pi/2)}{-\sin(-\pi/2)} + \frac{\sin(\pi/2)}{-\sin(\pi/2)}\right] && (144)\\
&= 2\pi i[-1 + (-1)] = -4\pi i && (145)
\end{aligned}$$

Example Evaluate $\oint_c e^{3/z} dz$, where the contour C is the circle $|z| = 1$.
Solution: $z = 0$ is an essential singularity of the integrand $f(z) = e^{3/z}$ and so neither of the two above procedure are applicable to find the residue of f at that point. Nevertheless, we saw demonstrate above that

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \cdots \qquad (146)$$

i.e. $Res(f(z), 0) = a_{-1} = 3$. From,

$$\oint_c f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k) \qquad (147)$$

where z_k are the isolate singularities of f, we get

$$\oint_c e^{3/z} dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k) = 2\pi i Res(f(z), 0) = 6\pi i \qquad (148)$$

Some Consequences of the Residue Theorem

Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$ The basic idea here is to convert a real trigonometric integral into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin.

We begin by parametrizing the contour by $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, then

$$dz = ie^{i\theta}d\theta \tag{149}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{150}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{151}$$

or

$$d\theta = \frac{dz}{iz} \tag{152}$$

$$\cos\theta = \frac{1}{2}(z + z^{-1}) \tag{153}$$

$$\sin\theta = \frac{1}{2i}(z - z^{-1}) \tag{154}$$

then

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta \to \oint_C F(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1}))\frac{dz}{iz} \tag{155}$$

where C is the unit circle $|z| = 1$.

Example Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}d\theta$
Solution: using the above substitutions we get

$$\oint_C \frac{1}{[2 + \frac{1}{2}(z + z^{-1})]^2}\frac{dz}{iz} = \oint_C \frac{1}{(2 + \frac{z^2+1}{2z})^2}\frac{dz}{iz} \tag{156}$$

$$= \frac{4}{i}\oint_C \frac{z}{(z^2 + 4z + 1)^2}dz \tag{157}$$

$$= \frac{4}{i}\oint_C \frac{z}{[(z - z_1)(z - z_2)]^2}dz \tag{158}$$

with $z_1 = -2 - \sqrt{3}$ and $z_2 = -2 + \sqrt{3}$. Because only z_2 is inside the unit circle C, we have

$$\oint_C \frac{1}{[2 + \frac{1}{2}(z + z^{-1})]^2}\frac{dz}{iz} = \frac{4}{i}\oint_C \frac{z}{[(z - z_1)(z - z_2)]^2}dz \tag{159}$$

$$= \frac{4}{i}\oint_C \frac{z}{(z - z_1)^2(z - z_2)^2}dz \tag{160}$$

$$= \frac{4}{i}2\pi i Res(f(z), z_2) \tag{161}$$

where

$$
\begin{aligned}
Res(f(z),z_2) &= \lim_{z\to z_2}\frac{d}{dz}(z-z_2)^2 f(z) && (162)\\
&= \lim_{z\to z_2}\frac{d}{dz}(z-z_2)^2\frac{z}{(z-z_1)^2(z-z_2)^2} && (163)\\
&= \lim_{z\to z_2}\frac{d}{dz}\frac{z}{(z-z_1)^2} && (164)\\
&= \lim_{z\to z_2}\frac{-z-z_1}{(z-z_1)^3}=\frac{1}{6\sqrt{3}} && (165)
\end{aligned}
$$

then

$$
\begin{aligned}
\oint_C \frac{1}{[2+\frac{1}{2}(z+z^{-1})]^2}\frac{dz}{iz} &= \frac{4}{i}2\pi i Res(f(z),z_2) && (166)\\
&= \frac{4}{i}2\pi i\frac{1}{6\sqrt{3}} && (167)\\
&= \frac{4\pi}{3\sqrt{3}} && (168)
\end{aligned}
$$

and, finally,

$$\int_0^{2\pi}\frac{1}{(2+\cos\theta)^2}d\theta=\frac{4\pi}{3\sqrt{3}} \tag{169}$$

Evaluation of Real Improper Integrals

Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$ Suppose $y=f(x)$ is a real function that is defined and continuous on the interval $[0,\infty)$ defined as

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} f(x)dx+\int_{0}^{\infty} f(x)dx = I_1+I_2 \tag{170}$$

with

$$
\begin{aligned}
I_1 &= \int_0^{\infty} f(x)dx=\lim_{R\to\infty}\int_0^R f(x)dx && (171)\\
I_2 &= \int_{-\infty}^{0} f(x)dx=\lim_{R\to-\infty}\int_{-R}^0 f(x)dx && (172)
\end{aligned}
$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2 , is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent. It is important to remember that the above definition for the improper integral is not the same as $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$.

For the integral $\int_{-\infty}^{\infty} f(x)dx$ to be convergent, the limits $\lim_{R\to\infty}\int_0^R f(x)dx$ and $\lim_{R\to-\infty}\int_{-R}^0 f(x)dx$ must exist independently of one another. But, in the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by

$$\int_{-\infty}^{\infty} f(x)dx=\lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{173}$$

On the other hand, the symmetric limit $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

The limit in

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{174}$$

if it exists, is called the **Cauchy principal value (P.V.)** of the integral and is written

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx \tag{175}$$

Cauchy Principal Value When an integral of form $\int_{-\infty}^{\infty} f(x)dx$ converges, its Cauchy principal value is the same as the value of the integral. If the integral diverges, it may still possess a Cauchy principal value.

About even functions Suppose $f(x)$ is continuous on $(-\infty,\infty)$ and is an even function, that is, $f(-x)=f(x)$. If the Cauchy principal value exists,

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}P.V.\int_{-\infty}^{\infty} f(x)dx \tag{176}$$

$$\int_{-\infty}^{\infty} f(x)dx = P.V.\int_{-\infty}^{\infty} f(x)dx \tag{177}$$

About evaluation of the improper integral To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x)=p(x)/q(x)$ is continuous on $(-\infty,\infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R,R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z)=p(z)/q(z)$ in the upper half-plane $Im(z)>0$, Fig. 3. By Theorem 6.16 we have

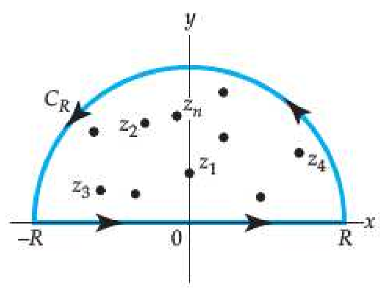

Figure 3: (From the book)

$$\begin{aligned}\oint_C f(z)dz &= \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx &(178)\\ &= 2\pi i\sum_{k=1}^{n} Res(f(z),z_k) &(179)\end{aligned}$$

where z_k , $k=1,2,\cdots,n$ denotes poles in the upper half-plane. If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R\to\infty$, then we have

$$\begin{aligned}P.V.\int_{-\infty}^{\infty} f(x)dx &= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx &(180)\\ &= 2\pi i\sum_{k=1}^{n} Res(f(z),z_k) &(181)\end{aligned}$$

Example Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.
Solution: let us write

$$f(z) = \frac{1}{(z^2+1)(z^2+9)} = \frac{1}{(z-i)(z+i)(z-3i)(z+3i)} \tag{182}$$

we take C be the closed contour consisting of the interval $[-R, R]$ on the x-axis and the semicircle C_R of radius $R > 3$.

$$\begin{aligned} \oint_C f(z)dz &= \int_{-R}^{R} f(z)dz + \int_{C_R} f(z)dz = I_1 + I_2 \qquad (183) \\ &= 2\pi i[Res(f(z), i) + Res(f(z), 3i)] = 2\pi i \left(\frac{1}{16i} - \frac{1}{48i} \right) \\ &= \frac{\pi}{12} \qquad (184) \end{aligned}$$

We now want to let $R \to \infty$. Before doing this, we use the following inequality valid in the contour C_R

$$|(z^2+1)(z^2+9)| \geq ||z^2| - 1| \cdot ||z^2| - 9| = (R^2 - 1)(R^2 - 9) \tag{185}$$

Since the length L of the semicircle is πR, it follows from the ML-inequality, Theorem 5.3, that

$$|I_2| = \left| \int_{C_R} \frac{1}{(z^2+1)(z^2+9)} \right| \leq \frac{\pi R}{(R^2-1)(R^2-9)} \tag{186}$$

Then, $|I_2| \to 0$ as $R \to \infty$, and so

$$\lim_{R\to\infty} I_2 = 0 \tag{187}$$

$$\tag{188}$$

and then,

$$\lim_{R\to\infty} I_1 = \frac{\pi}{12} \tag{189}$$

Finally,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)} dx = P.V. \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12} \tag{190}$$

Because the integrand $f(z)$ is an even function, the existence of the Cauchy principal value implies that the original integral converges to $\pi/12$, i.e.

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = \frac{\pi}{12} \tag{191}$$

Sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$ is always true are summarized in the next theorem.

Theorem (6.17): Behavior of Integral as $R \to \infty$ Suppose $f(z) = p(z)/q(z)$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

Example Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$.
Solution: The conditions given in Theorem 6.17 are satisfied. $f(z) = 1/(z^4+1)$ has simple poles in the upper half-plane at $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$, with residues

$$Res(f(z), z_1) = -\frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}} \tag{192}$$

$$Res(f(z), z_2) = \frac{1}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}} \tag{193}$$

then

$$PV \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i[Res(f(z), z_1) + Res(f(z), z_2)] = \frac{\pi}{\sqrt{2}} \tag{194}$$

Since the integrand $f(z)$ is an even function, the original integral converges to $\pi/\sqrt{2}$, i.e.

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = \frac{\pi}{\sqrt{2}} \tag{195}$$

Fourier Integrals: $\int_{-\infty}^{\infty} f(x) \cos \alpha x dx$ **and** $\int_{-\infty}^{\infty} f(x) \sin \alpha x dx$ Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$.

We can write

$$\int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx + i \int_{-\infty}^{\infty} f(x) \sin \alpha x dx \tag{196}$$

whenever both integrals on the right-hand side of converge. Suppose $f(x) = p(x)/q(x)$ is a rational function that is continuous on $(-\infty, \infty)$. Then both Fourier integrals in can be evaluated at the same time by considering the complex integral $\int_C f(z) e^{i\alpha z} dz$, where $\alpha > 0$, and the contour C consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.

The next theorem gives sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$.

Theorem (6.18): Behavior of Integral as $R \to \infty$ Suppose $f(z) = p(z)/q(z)$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} f(z) e^{i\alpha z} dz \to 0$ as $R \to \infty$.

Example Evaluate the Cauchy principal value of $\int_0^{\infty} \frac{x \sin x}{x^2+9} dx$.
Solution: First we rewrite the integral

$$\int_0^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx \tag{197}$$

With $\alpha = 1$ we build

$$\oint_C \frac{z}{z^2+9} e^{iz} dz \tag{198}$$

with C a semicircle in the upper complex plane. Using theorem 6.16

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx = 2\pi i Res(f(z) e^{iz}, 3i) \tag{199}$$

where

$$Res(f(z) e^{iz}, 3i) = Res(\frac{z}{z^2+9} e^{iz}, 3i) = \frac{z}{2z} e^{iz} \Big|_{z=3i} = \frac{e^{-3}}{2} \tag{200}$$

The integral in the contour C_R goes to zero, then

$$PV\int_{-\infty}^{\infty}\frac{x}{x^2+9}e^{ix}dx = 2\pi i\left(\frac{e^{-3}}{2}\right) = i\frac{\pi}{e^3} \tag{201}$$

Then,

$$\int_{-\infty}^{\infty}\frac{x}{x^2+9}e^{ix}dx = \int_{-\infty}^{\infty}\frac{x\cos x}{x^2+9}dx + i\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+9}dx = i\frac{\pi}{e^3} \tag{202}$$

Equating real and imaginary parts we get

$$PV\int_{-\infty}^{\infty}\frac{x\cos x}{x^2+9}dx = 0 \tag{203}$$

$$PV\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+9}dx = \frac{\pi}{e^3} \tag{204}$$

Finally, in view of the fact that the integrand is an even function, we obtain the value of the required integral,

$$\int_{0}^{\infty}\frac{x\sin x}{x^2+9}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+9}dx = \frac{\pi}{2e^3} \tag{205}$$

Indented Contours In the situation where f has poles on the real axis, we must use an **indented(mellado) contour** as illustrated in Figure 4. The symbol C_r denotes a semicircular contour centered at $z = c$ and oriented in the positive direction. The next theorem is important to this discussion.

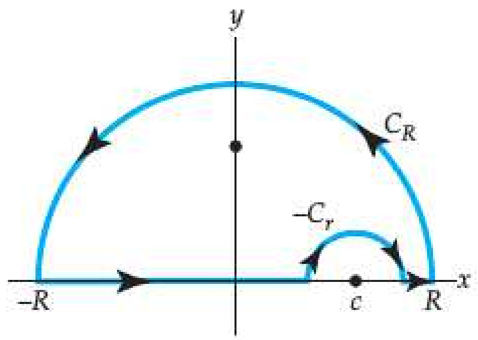

Figure 4: (From the book)

Theorem (6.19): Integral of functions with pole on the real axis Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \le \theta \le \pi$, then

$$\lim_{r\to 0}\int_{C_r} f(z)dz = \pi i Res(f(z), c) \tag{206}$$

Proof: See pag. 359 in the book

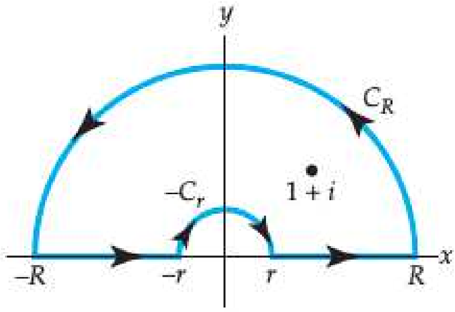

Figure 5: (From the book)

Example Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}dx$.
Solution: Let us consider the contour integral

$$\oint_C \frac{e^{iz}}{z(z^2-2z+2)}dz \tag{207}$$

The function $f(z) = \frac{e^{iz}}{z(z^2-2z+2)}$ has a pole at $z=0$ and at $z=1+i$ in the upper half-plane. The contour C, shown in Figure 5, is indented at the origin, then

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i Res(f(z)e^{iz}, 1+i) \tag{208}$$

By taking the limits $R \to \infty$ and $r \to 0$, it follows from Theorems 6.18 and 6.19 that

$$PV\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx - \pi i Res(f(z)e^{iz}, 0) = 2\pi i Res(f(z)e^{iz}, 1+i)$$

where

$$Res(f(z)e^{iz}, 0) = \frac{1}{2} \tag{209}$$

$$Res(f(z)e^{iz}, 1+i) = -\frac{e^{-1+i}}{4}(1+i) \tag{210}$$

then

$$PV\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx = \pi i\frac{1}{2} + 2\pi i\left(-\frac{e^{-1+i}}{4}(1+i)\right) \tag{211}$$

Using $e^{-1+i} = e^{-1}(\cos 1 + i\sin 1)$ and equating real and imaginary parts, we get

$$PV\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)}dx = \frac{\pi}{2}e^{-1}(\sin 1 + \cos 1) \tag{212}$$

$$PV\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}dx = \frac{\pi}{2}[1 + e^{-1}(\sin 1 - \cos 1)] \tag{213}$$

Integration along a Branch Cut

Branch Point at $z = 0$ Here we examine integrals of the form $\int_0^\infty f(x)dx$, where the integrand $f(x)$ is algebraic but when it is converted to a complex function, the resulting integrand $f(z)$ has, in addition to poles, a *nonisolated singularity at* $z=0$.

Example: Integration along a Branch Cut Evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx$.
Solution: The above real integral is improper for two reasons: (i) an infinite discontinuity at $x = 0$ and (ii) the infinite limit of integration. Moreover, it can be argued from the facts that the integrand behaves like $x^{-1/2}$ near the origin and like $x^{-3/2}$ as $x \to \infty$, that the integral converges.

We form the integral

$$\oint_C \frac{1}{z^{1/2}(z+1)}dz \tag{214}$$

where C is the closed contour shown in Figure 6 consisting of four components. The integrand

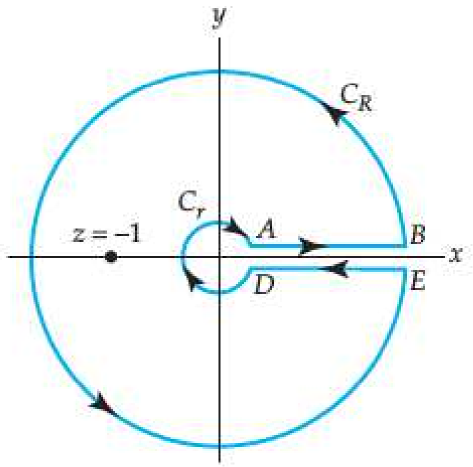

Figure 6: (From the book)

$f(z)$ of the contour integral is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{i\pi}$. Hence we can write

$$\oint_C \frac{1}{z^{1/2}(z+1)}dz = 2\pi i Res(f(z), -1) \tag{215}$$

$$\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i Res(f(z), -1) \tag{216}$$

$$\tag{217}$$

with $f(z) = \frac{1}{z^{1/2}(z+1)}$. The segment AB coincides with the upper side of the positive real axis for which $\theta = 0$, $z = xe^{0i}$; while, the segment ED coincides with the lower side of the positive real axis for which $\theta = 2\pi$, $z = xe^{(0+2\pi)i}$, then

$$\int_{ED} = \int_R^r \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1}(e^{2\pi i}dx) \tag{218}$$

$$= -\int_R^r \frac{x^{-1/2}}{x+1}dx \tag{219}$$

$$= \int_r^R \frac{x^{-1/2}}{x+1}dx \tag{220}$$

$$\int_{AB} = \int_r^R \frac{(xe^{0i})^{-1/2}}{xe^{0i}+1}(e^{0i}dx) \tag{221}$$

$$= \int_r^R \frac{x^{-1/2}}{x+1}dx \tag{222}$$

Now with $z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R , respectively, it can be shown that

$$\int_{C_r} \to 0 \tag{223}$$

$$\int_{C_R} \to 0 \tag{224}$$

as $r \to 0$ and $R \to \infty$, respectively. Then

$$\lim_{r\to 0, R\to\infty} \left[\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} \right] = 2\pi i Res(f(z), -1) \tag{225}$$

is the same as

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx = 2\pi i Res(f(z), -1) \tag{226}$$

with

$$Res(f(z), -1) = z^{-1/2}\big|_{z=e^{i\pi}} = e^{-\pi i/2} = -i \tag{227}$$

then

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)} dx = \pi \tag{228}$$

The Argument Principle and Rouché's Theorem

Argument Principle Unlike the foregoing discussion in which the focus was on the evaluation of real integrals, we next apply residue theory to the location of zeros of an analytic function.

Theorem (6.20): Argument Principle Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C. Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p \tag{229}$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p , zeros and poles are counted according to their order or multiplicities.
Proof: See pag. 363 in the book.

Theorem (6.21): Rouché's Theorem Let C be a simple closed contour lying entirely within a domain D. Suppose f and g are analytic in D. If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C, then f and g have the same number of zeros (counted according to their order or multiplicities) inside C.
Proof: See pag. 365 in the book.

The Rouché's Theorem is helpful in determining the number of zeros of an analytic function (in a given region).