Integración en el plano complejo

Credit: This notes are 100% from chapter 5 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

Real Integrals

Terminology Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the initial and terminal points of C, that is, $(x(a), y(a))$ and $(x(b), y(b))$, be denoted by the symbols A and B, respectively. We say that:
(i) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).
(ii) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \cdots, C_n$ joined end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.
(iii) C is a simple curve if the curve C does not cross itself except possibly at $t = a$ and $t = b$.
(iv) C is a closed curve if $A = B$.
(v) C is a simple closed curve if the curve C does not cross itself and $A = B$; that is, C is simple and closed.

Method of Evaluation–C Defined Parametrically The line integrals can be evaluated in two ways, depending on whether the curve C is defined by a pair of parametric equations or by an explicit function. Either way, *the basic idea is to convert a line integral to a definite integral* in a single variable. If C is smooth curve parametrized by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, then replace x and y in the integral by the functions $x(t)$ and $y(t)$, and the appropriate differential dx, dy, or ds by

$$dx = x'(t)dt \quad (1)$$
$$dy = y'(t)dt \quad (2)$$
$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}dt \quad (3)$$

In this manner each of the line integrals becomes a definite integral in which the variable of integration is the parameter t. That is,

$$\int_C G(x,y)dx = \int_a^b G(x(t),y(t))x'(t)dt \tag{4}$$

$$\int_C G(x,y)dy = \int_a^b G(x(t),y(t))y'(t)dt \tag{5}$$

$$\int_C G(x,y)ds = \int_a^b G(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}dt \tag{6}$$

Method of Evaluation–C Defined by a Function If the path of integration C is the graph of an explicit function $y = f(x)$, $a \le x \le b$, then we can use x as a parameter. In this situation, $dy = f'(x)dx$, and the differential $ds = \sqrt{1+[f'(x)]^2}dx$. Then,

$$\int_C G(x,y)dx = \int_a^b G(x(t),f(x))dx \tag{7}$$

$$\int_C G(x,y)dy = \int_a^b G(x(t),f(x))f'(x)dx \tag{8}$$

$$\int_C G(x,y)ds = \int_a^b G(x(t),f(x))\sqrt{1+[f'(x)]^2}dx \tag{9}$$

A line integral along a piecewise smooth curve C is defined as the sum of the integrals over the various smooth curves whose union comprises C.

It is important to be aware that a line integral is independent of the parametrization of the curve C, provided C is given the same orientation by all sets of parametric equations defining the curve.

Complex Integrals

Curves Revisited Suppose the continuous real-valued functions $x = x(t)$, $y = y(t)$, $a \le t \le b$, are parametric equations of a curve C in the complex plane. If we use these equations as the real and imaginary parts in $z = x + iy$, we can describe the points z on C by means of a complex-valued function of a real variable t called a **parametrization** of C:

$$z(t) = x(t) + iy(t) \tag{10}$$

with $a \le t \le b$. For example, the parametric equations $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, describe a unit circle centered at the origin. A parametrization of this circle is $z(t) = \cos t + i \sin t$, or $z(t) = e^{it}$, $0 \le t \le 2\pi$.

The point $z(a/b) = x(a/b)+iy(a/b)$ or $A/B = (x(a/b), y(a/b))$ is called the **initial/terminal point** of C. $z(t) = x(t) + iy(t)$ could also be interpreted as a two-dimensional vector function, with $z(a)$ and $z(b)$ being as position vectors. As t varies from $t = a$ to $t = b$ we can envision the curve C being traced out by the moving arrowhead of $z(t)$.

Contours The notions of curves in the complex plane that are smooth, piecewise smooth, simple, closed, and simple closed are easily formulated in terms of the vector function $z(t) = x(t) + iy(t)$. Suppose that its derivative is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \le t \le b$. The vector $z(t)$ is tangent to C at P. In other words, a smooth curve can have no sharp corners or cusps. A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \cdots, C_n$ are joined together. A curve C in the complex plane is said to be a **simple** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$. C is a **closed curve** if $z(a) = z(b)$. C is a **simple closed curve** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$. In complex analysis, a piecewise smooth curve C is called a **contour** or path.

We define the **positive direction/orientation** on a contour C to be the direction on the curve corresponding to increasing values of the parameter t. In the case of a simple closed curve C, the positive direction roughly corresponds to the counterclockwise direction. The **negative direction** is the direction opposite the positive direction.

Complex or Contour Integral An integral of a function f of a complex variable z that is defined on a contour C is denoted by $\int_C f(z)dz$ and is called a **complex or contour integral**,

$$\int_C f(z)dz = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \tag{11}$$

If the limit exists, then f is said to be integrable on C. The limit exists whenever if f is continuous at all points on C and C is either smooth or piecewise smooth. Consequently we shall, hereafter, assume these conditions as a matter of course. Moreover, we will use the notation $\oint_C f(z)dz$ to represent a complex integral around a positively oriented closed curve.

By writing $f = u + iv$ and $\Delta z = \Delta x + i\Delta y$ we can write, in a short hand notation

$$\begin{aligned} \int_C f(z)dz &= \lim \sum (u + iv)(\Delta x + i\Delta y) && (12) \\ &= \lim \left[\sum (u\Delta x - v\Delta y) + i \sum (v\Delta x + u\Delta y) \right] && (13) \end{aligned}$$

The interpretation of the last line is

$$\int_C f(z)dz = \int_C udx - vdy + i \int_C vdx + udy \tag{14}$$

If $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ are parametric equations of C, then $dx = x'(t)dt$, $dy = y'(t)dt$, then

$$\begin{aligned} \int_C udx - vdy + i \int_C vdx + udy &= \\ \int_a^b [u(x(t), y(t))\, x'(t) - v(x(t), y(t))\, y'(t)]dt & \\ +i \int_a^b [v(x(t), y(t))\, x'(t) + u(x(t), y(t))\, y'(t)]dt & && (15) \end{aligned}$$

If we use the complex-valued function $z(t) = x(t) + iy(t)$ to describe the contour C, then Eq. (15) is the same as $\int_a^b f(z(t))\, z'(t)dt$ when the integrand

$$f(z(t))\, z'(t) = [u(x(t), y(t)) + iv(x(t), y(t))][x'(t) + iy'(t)] \tag{16}$$

is multiplied out and $\int_a^b f(z(t))\, z'(t)dt$ is expressed in terms of its real and imaginary parts. Thus we arrive at a practical means of evaluating a contour integral.

Evaluation of a Contour Integral If f is continuous on a smooth curve C given by the parametrization $z(t) = x(t) + iy(t)$, $a \le t \le b$, then

$$\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)\, dt \tag{17}$$

Example Evaluate the contour integral $\int_C \bar{z}dz$, where C is given by $x = 3t$, $y = t^2$, $-1 \le t \le 4$.
Solution:
$z(t) = 3t + it^2$, $z'(t) = 3 + i2t$ and $f(z(t)) = 3t - it^2$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt \tag{18}$$

$$\int_C \bar{z}dz = \int_a^b (3t - it^2)(3 + i2t)dt = 195 + i65 \tag{19}$$

Example For some curves the real variable x itself can be used as the parameter. For example, to evaluate $\int_C (8x^2 - iy)dz$ on the line segment $y = 5x$, $0 \le x \le 2$, we write $z = x + iy = x + 5xi$ (i.e. $y = 5x$), $dz = (1 + 5i)dx$, then

$$\int_C (8x^2 - iy)dz = \int_0^2 (8x^2 - i5x)(1 + 5i)dx = \frac{214}{3} + i\frac{290}{3} \tag{20}$$

Properties(Theorem 5.2) Suppose the functions f and g are continuous in a domain D, and C is a smooth curve lying entirely in D. Then

(i) $\int_C kf(z)dz = k\int_C f(z)dz$, k a complex constant.

(ii) $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$.

(iii) $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)dz = -\int_C f(z)dz$, where $-C$ denotes the curve having the opposite orientation of C.

All these four properties hold if C is a piecewise smooth curve in D.

Theorem (5.3): A Bounding Theorem or ML-inequality If f is continuous on a smooth curve C and if $|f(z)| \le M$ for all z on C, then

$$|\int_C f(z)dz| \le ML \tag{21}$$

where L is the length of C , i.e. $L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_a^b |z'(t)|dt$, where $z'(t) = x'(t) + iy'(t)$.

It follows that since f is continuous on the contour C, the bound M for the values $f(z)$ in Theorem 5.3 will always exist.

Example Find an upper bound for the absolute value of $\oint_C e^z(z+1)^{-1}dz$ where C is the circle $|z| = 4$.
Solution:
The length L of the circle is 8π. Next, for all points z on the circle $|z+1| \geq |z| - 1 = 3$. Thus

$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{e^x}{3} \leq \frac{e^4}{3} \tag{22}$$

where we used that on the circle $|z| = 4 \Rightarrow \max x = 4$, then

$$\left|\int_C f(z)dz\right| \leq ML = \frac{8\pi e^4}{3} \tag{23}$$

Cauchy-Goursat Theorem

In this section we shall concentrate on contour integrals, where the contour C is a simple closed curve with a positive (counterclockwise) orientation. Specifically, we shall see that when f is analytic in a special kind of domain D, the value of the contour integral$\oint_C f(z)dz$ is the same for any simple closed curve C that lies entirely within D.

Simply and Multiply Connected Domains. A **domain** is an open connected set in the complex plane. We say that a domain D is **simply connected** if every simple closed contour C lying entirely in D can be shrunk(encogido) to a point without leaving D. A simply connected domain has no "holes" in it. The entire complex plane is an example of a simply connected domain; the annulus defined by $1 < |z| < 2$ is not simply connected. A domain that is not simply connected is called a **multiply connected domain**; that is, a multiply connected domain has "holes" in it.

In 1825 the French mathematician Louis-Augustin Cauchy proved one of the most important theorems in complex analysis:

Cauchy's Theorem Suppose that a function f is analytic in a simply connected domain D and that f is continuous in D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0 \tag{24}$$

Proof: See pag. 257 in the book.

In 1883 the French mathematician Edouard Goursat proved that the assumption of continuity of f is not necessary to reach the conclusion of Cauchy's theorem. The resulting modified version of Cauchy's theorem is known today as the Cauchy-Goursat theorem:

Theorem (5.4): Cauchy-Goursat Theorem Suppose that a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0 \tag{25}$$

Proof: See Appendix II in the book.

Since the interior of a simple closed contour is a simply connected domain, the Cauchy-Goursat theorem can be stated in the slightly more practical manner:
If f is analytic at all points within and on a simple closed contour C, then $\oint_C f(z)dz = 0$.

Example Using an arbitrary shaped contour C in the first quadrant calculates $\oint_C e^z dz$.
Solution: The function $f(z) = e^z$ is entire and consequently is analytic at all points within and on the simple closed contour C. It follows that $\oint_C e^z dz = 0$. The point in this example is that $\oint_C e^z dz = 0$ for *any* simple closed contour in the complex plane. Indeed, it follows that for any simple closed contour C and any entire function f that the integral is nil, for example

$$\oint_C \sin z dz = 0 \tag{26}$$

$$\oint_C \cos z dz = 0 \tag{27}$$

$$\oint_C \sum_{k=0}^{n} a_k z^k dz = 0 \tag{28}$$

and so on.

Example Evaluate $\oint_C \frac{dz}{z^2}$, where the contour C is the ellipse $(x-2)^2 + \frac{1}{4}(y-5)^2 = 1$.
Solution: The rational function $f(z) = 1/z^2$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the simple closed elliptical contour C. Thus, $\oint_C \frac{dz}{z^2} = 0$.

Principle of deformation of contours If f is analytic in a multiply connected domain D then we cannot conclude that $\oint_C f(z)dz = 0$ for every simple closed contour C in D. To begin, suppose that D is a doubly connected domain (i.e. a domain with a single "hole") and C and C_1 are simple closed contours such that C_1 surrounds the "hole" in the domain and is interior to C (see Fig. 1(a)). Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1. By introducing the crosscut AB shown in Figure 1(b), the region bounded between the curves is now simply connected. From (iv) of Theorem 5.2, the integral from A to B has the opposite value of the integral from B to A, then

$$0 = \oint_C f(z)dz + \int_{AB} f(z)dz + \int_{-AB} f(z)dz + \oint_{C_1} f(z)dz$$

(aquí C se recorre en sentido antihorario y C_1 en sentido horario)
luego

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz \tag{29}$$

(aquí ambos, C y C_1, se recorren en sentido antihorario)
This result is sometimes called the principle of deformation of contours since we can think of the contour C_1 as a continuous deformation of the contour C. Under this deformation of contours, the value of the integral does not change. Then, whe can evaluate an integral over a complicated simple closed contour C by replacing it with a contour C_1 that is more convenient.

The next theorem summarizes the general result for a multiply connected domain with n "holes."

Theorem (5.5): Cauchy-Goursat Theorem for Multiply Connected Domains Suppose $C, C_1, \cdots, C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \cdots, C_n$ are interior to C but the regions interior to each C_k , $k = 1, 2, \cdots, n$, have no points in common.

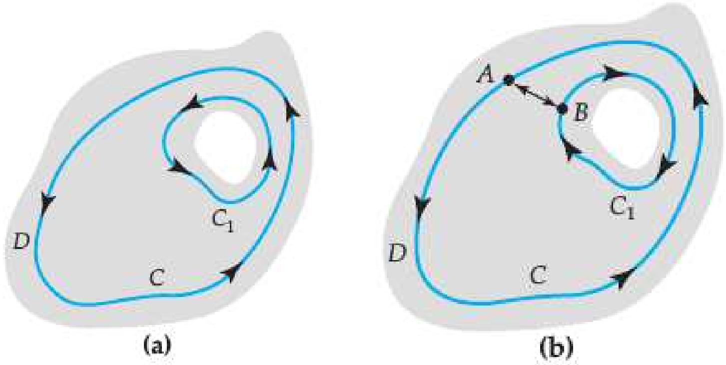

Figure 1: Doubly connected domain D (from the book)

If f is analytic on each contour and at each point interior to C but exterior to all the C_k , $k = 1, 2, \cdots, n$, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz \tag{30}$$

Example Evaluate $\oint_C \frac{dz}{z-i}$, where C is a complicated contour which contains $z = i$.
Solution: we choose the more convenient circular contour C_1 centered at $z_0 = i$ and radius $r = 1$, i.e. $|z - i| = 1$. It can be parametrized by $z = i + e^{it}$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\oint_C \frac{dz}{z-i} &= \oint_{C_1} \frac{dz}{z-i} = \int_0^{2\pi} \frac{dz}{e^{it}} = \int_0^{2\pi} \frac{ie^{it}dt}{e^{it}} \\ &= 2\pi i \end{aligned} \tag{31}$$

This result can be generalized: if z_0 is any constant complex number interior to any simple closed contour C, then for n an integer we have

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases} \tag{32}$$

The fact that this integral is zero when $n \neq 1$ follows only partially from the Cauchy-Goursat theorem. When n is zero or a negative integer, then $1/(z - z_0)^n$ is a polynomial and therefore entire. Theorem 5.4 then indicates that the integral is zero.

Analyticity of the function f at all points within and on a simple closed contour C is *sufficient* to guarantee that $\oint_C f(z)dz = 0$. However, the previous example emphasizes that analyticity is not *necessary*.

Example Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}dz$, where C is the circle $|z - 2| = 2$.
Solution: The roots of the denominators are 1 and -3. The integrand fails at these roots. Of these two points, only $z = 1$ lies within the contour C. Separating the roots by partial fraction

$$\frac{5z+7}{z^2+2z-3} = \frac{3}{z-1} + \frac{2}{z+3} \tag{33}$$

we have

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = \oint_C \frac{3}{z-1}dz + \oint_C \frac{2}{z+3}dz \tag{34}$$

from the above calculation, the first integral gives $2\pi i$, whereas the second gives 0 by the Cauchy-Goursat theorem. Then

$$\oint_C \frac{5z+7}{z^2+2z-3}dz = 3(2\pi i) + 2(0) = 6\pi i \tag{35}$$

Example Evaluate $\oint_C \frac{dz}{z^2+1}$, where C is the circle $|z| = 4$.
Solution: In this case the denominator of the integrand factors as $z^2 + 1 = (z - i)(z + i)$. Consequently, the integrand $1/(z^2 + 1)$ is not analytic at $z = \pm i$. Both of these points lie within the contour C. Using partial fraction decomposition once more, we have

$$\oint_C \frac{dz}{z^2+1} = \oint_C \frac{dz}{2i(z-i)} - \oint_C \frac{dz}{2i(z+i)} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

Next we surround the points $z = i$ and $z = -i$ by circular contours C_1 and C_2 , respectively, that lie entirely within C. Specifically, the choice $|z - i| = 1/2$ for C_1 and $|z + i| = 1/2$ for C_2 will suffice. Then

$$\begin{aligned}\oint_C \frac{dz}{z^2+1} &= \frac{1}{2i}\oint_{C_1} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz + \frac{1}{2i}\oint_{C_2} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz \\ &= \frac{1}{2i}[2\pi i - 0] + \frac{1}{2i}[0 - 2\pi i] = 0 \end{aligned} \tag{36}$$

Remark Throughout the foregoing discussion we assumed that C was a simple closed contour. It can be shown that the Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D.

There exist integrals $\int_C Pdx + Qdy$ whose value depends only on the initial point A and terminal point B of the curve C, and not on C itself. In this case we say that the line integral is **independent of the path**.

Independence of the Path Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be independent of the path if its value is the same for all contours C in D with initial point z_0 and terminal point z_1.

Theorem (5.6): Analyticity Implies Path Independence. Suppose that a function f is analytic in a simply connected domain D and C is any contour in D. Then $\int_C f(z)dz$ is independent of the path C.

Suppose, as shown in Figure 2 that C and C_1 are two contours lying entirely in a simply connected domain D and both with initial point z_0 and terminal point z_1. If f is analytic in D, it follows from the Cauchy-Goursat theorem that

$$[h!] \int_C f(z)dz + \int_{-C_1} f(z)dz = 0 \tag{37}$$

$$\int_C f(z)dz = \int_{C_1} f(z)dz \tag{38}$$

A contour integral $\int_C f(z)dz$ that is independent of the path C is usually written $\int_{z_0}^{z_1} f(z)dz$, where z_0 and z_1 are the initial and terminal points of C.

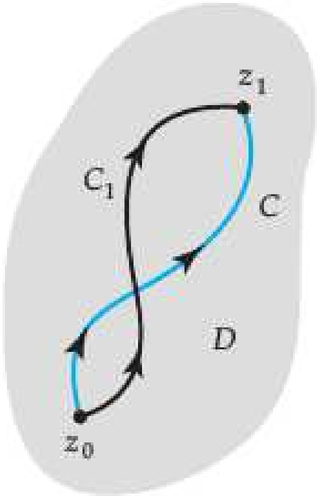

Figure 2: If f is analytic in D, integrals on C and C_1 are equal (from the book).

Example Evaluate $\int_C 2zdz$, where C is the contour shown in color in Figure 3.
Solution: Since the function $f(z) = 2z$ is entire, we can, in view of Theorem 5.6, replace the piecewise smooth path C by any convenient contour C_1 joining $z_0 = -1$ and $z_1 = -1+i$. Using the black contour in Fig. 3, then $z = -1 + iy$, $dz = idy$, $0 \le y \le 1$. Therefore,

$$\begin{aligned} \int_C 2zdz &= \int_{C_1} 2zdz = \int_0^1 2(-1+iy)(idy) \\ &= -2i\int_0^1 dy - 2\int_0^1 ydy = [-2i] - 2[\frac{1}{2}] = -1-2i \end{aligned}$$

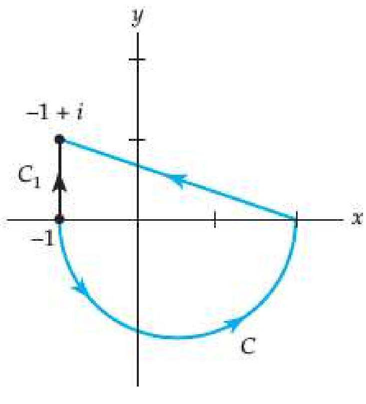

Figure 3: Alternative contour for the integral $\int_C 2zdz$ (from the book).

Antiderivative Suppose that a function f is continuous on a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an antiderivative of f. For example, the function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$.

Indefinite integral The most general antiderivative, or indefinite integral, of a function $f(z)$ is written $\int f(z)dz = F(z) + C$, where $F'(z) = f(z)$ and C is some complex constant. For example, $\int \sin z dz = -\cos z + C$.

Since an antiderivative F of a function f has a derivative at each point in a domain D, it is necessarily analytic and hence continuous at each point in D.

Theorem (5.7): Fundamental Theorem for Contour Integrals Suppose that a function f is continuous on a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0) \tag{39}$$

Example Calculate the integral $\int_C 2zdz$ with the same contour as in the previous example.
Solution: Now since the $f(z) = 2z$ is an entire function, it is continuous. Moreover, $F(z) = z^2$ is an antiderivative of f.

$$\int_{-1}^{-1+i} 2zdz = z^2\Big|_{-1}^{-1+i} = -1 - 2i \tag{40}$$

Example Evaluate $\int_C \cos z dz$, where C is any contour with initial point $z_0 = 0$ and terminal point $z_1 = 2 + i$.
Solution: $F(z) = \sin z$ is an antiderivative of $f(z) = \cos z$ since $F'(z) = \cos z = f(z)$. Therefore,

$$\int_C \cos z dz = \int_0^{2+i} \cos z dz = \sin z|_0^{2+i} = \sin(2+i) \approx 1.4031 - i0.4891 \tag{41}$$

Some Conclusions We can draw several immediate conclusions from Theorem 5.7.
(i) If the contour C is closed, then $z_0 = z_1$ and, consequently, $\int_C f(z)dz = 0$.
(ii) Since the value of $\int_C f(z)dz$ depends only on the points z_0 and z_1 , this value is the same for any contour C in D connecting these points, i.e. if a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.
(iii) If f is continuous and $\int_C f(z)dz$ is independent of the path C in a domain D, then f has an antiderivative everywhere in D.

If f is an analytic function in a simply connected domain D, it is necessarily continuous throughout D. This fact, when put together with the results in Theorem 5.6 (iii), leads to a theorem which states that an analytic function possesses an analytic antiderivative.

Theorem (5.8): Existence of an Antiderivative Suppose that a function f is analytic in a simply connected domain D. Then f has an antiderivative in D; that is, there exists a function F such that $F'(z) = f(z)$ for all z in D.

About the antiderivate of $1/z$ We saw for $|z| > 0$, $-\pi < arg(z) < \pi$, that $1/z$ is the derivative of Lnz. This means that under some circumstances Lnz is an antiderivative of $1/z$. But care must be exercised in using this result. For example, suppose D is the entire complex plane without the origin. The function $1/z$ is analytic in this multiply connected domain. If C is any simple closed contour containing the origin, it does not follow that $\oint_C dz/z = 0$. In fact, from the result for $\oint_c dz/(z - z_0)^n$ for $n = 1$ and $z_0 = 0$, we have $\oint_C dz/z = 2\pi i$. In this case, Lnz is not an antiderivative of 1/z in D since Lnz is not analytic in D. Recall, Lnz fails to be analytic on the nonpositive real axis.

Example Evaluate $\int_C 1/z dz$, where C is a contour in the first quadrant starting at $z_0 = 3$ and ending at $z = 2i$.
Solution: Suppose that D is the simply connected domain defined by $x > 0$, $y > 0$, i.e. D is the first quadrant in the z-plane. In this case, Lnz is an antiderivative of $1/z$ since both these functions are analytic in D. Hence,

$$\int_3^{2i} \frac{1}{z} dz = Lnz|_3^{2i} = Ln2i - Ln3 \quad (42)$$

$$= (\ln 2 + i\frac{\pi}{2}) + (\ln 3) \approx -0.4055 + i1.5708 \quad (43)$$

Example Evaluate $\int_C 1/z^{1/2} dz$, where C is the line segment between $z_0 = i$ and $z_1 = 9$.
Solution: Throughout we take $f_1(z) = z^{1/2}$ to be the principal branch of the square root function. In the domain $|z| > 0$, $-\pi < arg(z) < \pi$, the function $f_1(z) = 1/z^{1/2} = z^{-1/2}$ is analytic and possesses the antiderivative $F(z) = 2z^{1/2}$. Hence,

$$\int_i^9 \frac{1}{z^{1/2}} dz = 2z^{1/2}\Big|_i^9 = 2\left[3 - \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\right] = (6 - \sqrt{2}) - i\sqrt{2} \quad (44)$$

Remarks .
(i) Integration by parts: Suppose f and g are analytic in a simply connected domain D. Then,

$$\int f(z)g'(z)dz = f(z)g(z) - \int g(z)f'(z)dz \quad (45)$$

(ii) In addition, if z_0 and z_1 are the initial and terminal points of a contour C lying entirely in D, then

$$\int_{z_0}^{z_1} f(z)g'(z)dz = f(z)g(z)|_{z_0}^{z_1} - \int_{z_0}^{z_1} g(z)f'(z)dz \quad (46)$$

(iii) In complex analysis there is no complex counterpart to the mean-value theorem $\int_a^b f(x)dx = f(c)(b-a)$ of real analysis, valid if f is continuous on the closed interval $[a, b]$, and c is a number in the open interval (a, b).

Cauchy's Integral Formulas and their Consequences

The most significant consequence of the Cauchy-Goursat theorem is the following result: *the value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.*

After establishing this proposition we shall use it to further show that: *an analytic function f in a simply connected domain possesses derivatives of all orders.*

Cauchy's Two Integral Formulas

If f is analytic in a simply connected domain D and z_0 is any point in D, the quotient $f(z)/(z - z_0)$ is not defined at z_0 and hence is not analytic in D. Therefore, we cannot conclude that the integral of $f(z)/(z - z_0)$ around a simple closed contour C that contains z_0 is zero by the Cauchy-Goursat theorem. Indeed, as we shall now see, the integral of $f(z)/(z - z_0)$ around C has the value $2\pi i f(z_0)$. The first of two remarkable formulas is known simply as the Cauchy integral formula.

Theorem (5.9): Cauchy's Integral Formula Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz \tag{47}$$

Proof: See pag. 273 in the book.

Because the symbol z represents a point on the contour C, the integral $f(z_0) = \frac{1}{2\pi i} \oint_C f(z)/(z - z_0)dz$ indicates that the values of an analytic function f at points z_0 inside a simple closed contour C are determined by the values of f on the contour C.

Cauchy's integral formula can be used to evaluate contour integrals. Since we often work problems without a simply connected domain explicitly defined, a more practical restatement of Theorem 5.9 is:

If f is analytic at all points within and on a single contour C, and z_0 is any point interior to C, then $f(z_0) = \frac{1}{2\pi i} \oint_C f(z)/(z - z_0)dz$.

Example Evaluate $\oint_C (z^2 - 4z + 4)/(z + i)dz$, where C is the circle $|z| = 2$.
Solution: First, we identify $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. Next, we observe that f is analytic at all points within and on the contour C. Thus, by the Cauchy integral formula we obtain

$$\oint_C \frac{z^2 - 4z + 4}{z + i} dz = 2\pi i f(-i) = \pi(-8 + i6) \tag{48}$$

Example Evaluate $\oint_C z/(z^2 + 9)dz$, where C is the circle $|z - 2i| = 4$.
Solution: The roots of denominator are $3i$ and $-3i$. We see that $3i$ is the only point within the closed contour C at which the integrand fails to be analytic. Then,

$$\oint_C \frac{z}{z^2 + 9} dz = \oint_C \frac{z}{(z - 3i)(z + 3i)} dz = \oint_C \frac{f(z)}{z - 3i} dz \tag{49}$$

with $f(z) = z/(z + 3i)$, then

$$\oint_C \frac{z}{z^2 + 9} dz = 2\pi i f(3i) = i\pi \tag{50}$$

We shall now build on Theorem 5.9 by using it to prove that the values of the derivatives $f^{(n)}(z_0)$, $n = 1, 2, 3, \cdots$ of an analytic function are also given by an integral formula. This second integral formula is known by the name Cauchy's integral formula for derivatives.

Theorem (5.10): Cauchy's Integral Formula for Derivatives Suppose that f is analytic in a simply connected domain D and C is any simple closed contour lying entirely within D. Then for any point z_0 within C,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \tag{51}$$

Proof: The demostration for $n = 1$ is given in pag. 275 of the book.

The Cauchy's integral formula for derivatives can be used to evaluate integrals.

Example Evaluate $\oint_C (z+1)/(z^4+2iz^3)dz$, where C is the circle $|z|=1$.
Solution: The integrand is not analytic at $z=0$ and $z=-2i$, but only $z=0$ lies within the closed contour. By writing the integral as

$$\oint_C \frac{z+1}{z^4+2iz^3}dz = \oint_C \frac{z+1}{(z+2i)(z-0)^3}dz \tag{52}$$

we can identify, $z_0=0$, $n=2$, and $f(z)=(z+1)/(z+2i)$. Then, $f''(z)=(2-4i)/(z+2i)^3$ and $f''(0)=(2i-1)/4i$,

$$\oint_C \frac{z+1}{z^4+2iz^3}dz = \frac{2\pi i}{2!}f''(0) = -\frac{\pi}{4}+i\frac{\pi}{2} \tag{53}$$

Example Evaluate $\int_C (z^3+3)/(z(z-i)^2)dz$, where C is shown in Fig. 4.
Solution: Although C is not a simple closed contour, we can think of it as the union of two

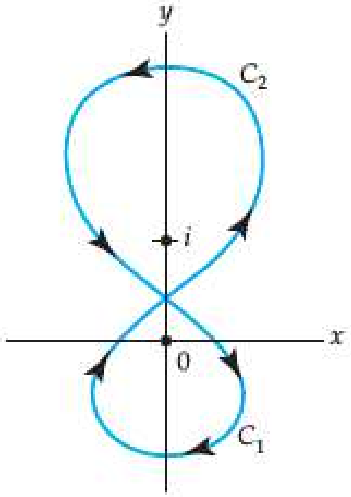

Figure 4: (from the book)

simple closed contours C_1 and C_2. Hence, we write

$$\int_C \frac{z^3+3}{z(z-i)^2}dz = \int_{C_1} \frac{z^3+3}{z(z-i)^2}dz + \int_{C_2} \frac{z^3+3}{z(z-i)^2}dz \tag{54}$$

$$= -\oint_{-C_1} \frac{z^3+3}{z(z-i)^2}dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}dz \tag{55}$$

$$= -\oint_{-C_1} \frac{f(z)}{z}dz + \oint_{C_2} \frac{g(z)}{(z-i)^2}dz \tag{56}$$

$$= -[2\pi i f(0)] + \frac{2\pi i}{1!}g'(i) \tag{57}$$

with

$$f(z) = \frac{z^3+3}{(z-i)^2} \tag{58}$$

$$g(z) = \frac{z^3+3}{z} \tag{59}$$

$$g'(z) = \frac{2z^3-3}{z^2} \tag{60}$$

and $f(0) = -3$, $g'(i) = 3 + 2i$, then

$$\int_C \frac{z^3+3}{z(z-i)^2}dz = -6\pi i + (-4\pi + 6\pi i) = -4\pi + i\,12\pi \tag{61}$$

Some Consequences of the Integral Formulas

Theorem (5.11): Derivative of an Analytic Function is Analytic Suppose that f is analytic in a simply connected domain D. Then f possesses derivatives of all orders at every point z in D. The derivatives $f', f'', f''', \cdots$ are analytic functions in D.

If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a simply connected domain D, from

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{62}$$

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \tag{63}$$

$$= \vdots \tag{64}$$

we can also conclude that the real functions u and v have continuous partial derivatives of all orders at a point of analyticity.

Theorem (5.12): Cauchy's Inequality Suppose that f is analytic in a simply connected domain D and C is a circle defined by $|z - z_0| = r$ that lies entirely in D. If $|f(z)| \leq M$ for all points z on C, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n} \tag{65}$$

Proof: Pag. 278 in the book

Theorem 5.12 is then used to prove the next theorem. The gist(esencia) of the theorem is that an entire function f, one that is analytic for all z, cannot be bounded unless f itself is a constant.

Theorem (5.13): Liouville's Theorem The only bounded entire functions are constants.
Proof: See pag. 279 of the book.

Theorem (5.14): Fundamental Theorem of Algebra If $p(z)$ is a nonconstant polynomial, then the equation $p(z) = 0$ has at least one root.
Proof: See pag. 279 of the book.

If $p(z)$ is a nonconstant polynomial of degree n, then $p(z) = 0$ has exactly n roots (counting multiple roots). See Problem 29 in Exercises 5.5 of the book.

Morera's Theorem gives a sufficient condition for analyticity. It is often taken to be the converse of the Cauchy-Goursat theorem.

Theorem (5.15): Morera's Theorem If f is continuous in a simply connected domain D and if $\oint_C f(z)dz = 0$ for every closed contour C in D, then f is analytic in D.
Proof: See pag. 280 of the book.

The next theorem tells us that $|f(z)|$ assumes its maximum value at some point z on the boundary C.

Theorem (5.16): Maximum Modulus Theorem Suppose that f is analytic and non-constant on a closed region R bounded by a simple closed curve C. Then the modulus $|f(z)|$ attains its maximum on C.

If the stipulation that $f(z) \neq 0$ for all z in R is added to the hypotheses of Theorem 5.16, then the modulus $|f(z)|$ also attains its minimum on C.

Example Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.
Solution: $|z| = z\bar{z}$, then for $z \to 2z + i5$ we get $|f(z)| = 4|z^2| + 20\Im(z) + 25$. Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. By Theorem 5.16, $max_{|z|\leq 2}|2z + 5i|$ occurs on the boundary $|z| = 2$. Therefore, on $|z| = 2$, $|2z + 5i| = \sqrt{41 + 20\Im(z)}$. This expression attains its maximum when $Im(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$. Thus, $max_{|z|\leq 2}|2z + 5i| = \sqrt{81} = 9$.

In this example, $f(z) = 0$ only at $z = -i5/2$ and that this point is outside the region defined by $|z| \leq 2$. Hence we can conclude that $|2z + 5i|$ attains its minimum when $Im(z)$ attains its minimum on $|z| = 2$ at $z = -2i$. Then, $min_{|z|\leq 2}|2z + 5i| = \sqrt{1} = 1$.