Distancia yAproximación

Credit: This notes are 100% from chapter 7 of the book entitled *Linear Algebra. A Modern Introduction* by David Poole. Thomson. Australia. 2006.

Introduction

By allowing ourselves to think of "distance" in a more flexible way, we will have the possibility of having a "distance" between polynomials, functions, matrices, and many other objects that arises in linear algebra.

Inner Product Spaces

Inner product: An inner product on a vector space V is an operation that assigns to every pair of vectors $\bar{u}$ and $\bar{v}$ in V a real number $\langle \bar{u}, \bar{v} \rangle$ such that the following properties hold for all vectors $\bar{u}$ and $\bar{v}$, and $\bar{w}$ in V and all scalars c:

1. $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$
2. $\langle \bar{u}, \bar{v} + \bar{w} \rangle = \langle \bar{u}, \bar{v} \rangle + \langle \bar{u}, \bar{w} \rangle$
3. $\langle c\bar{u}, \bar{v} \rangle = c\langle \bar{u}, \bar{v} \rangle$
4. $\langle \bar{u}, \bar{u} \rangle \geq 0$ and $\langle \bar{u}, \bar{u} \rangle = 0$ if and only if $\bar{u} = 0$

Inner product space: A vector space with an inner product is called an (real) inner product space.

Example 7.1:

- $\mathbb{R}^n$ is an inner product space with $\langle \bar{u}, \bar{v} \rangle = \bar{u} \cdot \bar{v} = \bar{u}^T \bar{v} = \sum_{i=1}^{n} u_i v_i$
- $\mathbb{R}^n$ is an inner product space with $\langle \bar{u}, \bar{v} \rangle = \sum_{i=1}^{n} \omega_i u_i v_i = \bar{u}^T W \bar{v}$ where $W = diag(\omega_i)$, with ω_i positive scalars. This is called weighted dot product.
- Let A be a symmetric, positive definite $n \times n$ matrix and $\bar{u}$ and $\bar{v}$ vectors in $\mathbb{R}^n$, the following defines an inner product: $\langle \bar{u}, \bar{v} \rangle = \bar{u}^T A \bar{v}$ (Example 7.3)
- In $\mathcal{P}_2$, let $p(x) = a_0 + a_1 x + a_2 x^2$ and $q(x) = b_0 + b_1 x + b_2 x^2$. The following product defines an inner product on $\mathcal{P}_2$: $\langle p(x), q(x) \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ (Example 7.4).
- Let f and g be in $\mathcal{C}[a, b]$, the vector space of all continuous functions on the closed interval $[a, b]$. The following product defines an inner product on $\mathcal{C}[a, b]$: $\langle f, g \rangle = \int_a^b f(x)g(x)dx$ (Example 7.5).

Exercise for the student in class (Example 7.2): Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vector in $\mathbb{R}^2$. Show that

$$\langle \bar{u}, \bar{v} \rangle = 2u_1v_1 + 3u_2v_2 \tag{1}$$

defines an inner product.

Properties of Inner Products

(Theorem 7.1) Let $\bar{u}$, $\bar{v}$, and $\bar{w}$ be vectors in an inner product space V and let c be a scalar,

a. $\langle \bar{u} + \bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$

b. $\langle \bar{u}, c\bar{v} \rangle = c\langle \bar{v}, \bar{u} \rangle$

c. $\langle \bar{u}, \bar{0} \rangle = \langle \bar{0}, \bar{v} \rangle = 0$

Proof: See book, pag. 544.

Length, Distance, and Orthogonality

Let $\bar{u}$ and $\bar{v}$ be vectors in an inner product space V,

Length-norm: The length or norm of $\bar{v}$ is $||\bar{v}|| = \sqrt{\langle \bar{v}, \bar{v} \rangle}$.

Distance: The distance between $\bar{u}$ and $\bar{v}$ is $\mathrm{d}(\bar{u}, \bar{v})$=$||\bar{u} - \bar{v}||$.

Orthogonal: $\bar{u}$ and $\bar{v}$ are orthogonal if $\langle \bar{u}, \bar{v} \rangle = 0$.

Exercise for the student in class (Example 7.6): Let f and g be in $\mathcal{C}[0,1]$, the vector space of all continuous functions on the closed interval $[0,1]$ with the following inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$, with $f(x) = x$ and $g(x) = 3x - 2$. Calculates

1. $||f||$
2. $\mathrm{d}(f, g)$
3. $\langle f, g \rangle$

Solution:

1. Let us first calculate

$$\langle f, f \rangle = \int_0^1 f^2(x)dx = \frac{1}{3} \tag{2}$$

then $||f|| = \sqrt{\langle f, f \rangle} = 1/\sqrt{3}$.

2. Since $\mathrm{d}(f, g)$=$||f - g|| = \sqrt{\langle (f-g), (f-g) \rangle}$

$$\langle (f-g), (f-g) \rangle = \int_0^1 (f-g)^2(x)dx = \frac{4}{3} \tag{3}$$

then $\mathrm{d}(f, g)$=$2/\sqrt{3}$.

3. Let us calculate $\langle f, g \rangle$

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx = 0 \tag{4}$$

Thus, f and g are orthogonal.

Pythagoras' Theorem (Theorem 7.2): Let $\bar{u}$ and $\bar{v}$ be vectors in an inner product space V. Then $\bar{u}$ and $\bar{v}$ are orthogonal if and only if

$$||\bar{u} + \bar{v}||^2 = ||\bar{u}||^2 + ||\bar{v}||^2 \tag{5}$$

Proof: In exercise 32 of the book it is asked to prove

$$||\bar{u} + \bar{v}||^2 = \langle \bar{u} + \bar{v}, \bar{u} + \bar{v} \rangle = ||\bar{u}||^2 + 2\langle \bar{u}, \bar{v} \rangle + ||\bar{v}||^2 \tag{6}$$

If follows immediately that $||\bar{u} + \bar{v}||^2 = ||\bar{u}||^2 + ||\bar{v}||^2$ if and only if $\langle \bar{u}, \bar{v} \rangle = 0$.

Orthogonal Projections and the Gram-Schmidt Process

Orthogonal set: an orthogonal set of vectors in an inner product space V is a set $\{\bar{v}_1, \cdots, \bar{v}_k\}$ of vectors from V such that $\langle \bar{v}_i, \bar{v}_j \rangle = 0$, whenever $\bar{v}_i \neq \bar{v}_j$.

Orthonormal set: and orthonormal set of vectors is an orthogonal set of unit vectors.

Orthogonal basis: an orthogonal basis for a subspace W of V is just a basis for W that is an orthogonal set.

Orthonormal basis: an orthonormal basis for a subspace W of V is a basis for W that is an orthonormal set.

Remark: In $\mathbb{R}^n$, the Gram-Schmidt Process(GSP) (Theorem 5.15 of the book) shows that every subspace has an orthogonal basis. We can mimic the construction of the GSP to show that every finite-dimensional subspace of an inner product space has an orthogonal basis–all we need to do is replace the dot product by the more general inner product. See next example.

Construction of an orthogonal basis (Example 7.8): Construct an orthogonal basis for $\mathcal{P}_2$ with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx \tag{7}$$

by applying the GSP to the basis $\{1, x, x^2\}$. Solution:
Let $\bar{x}_1 = 1$, $\bar{x}_2 = x$, and $\bar{x}_3 = x^2$. We begin by setting $\bar{v}_1 = \bar{x}_1 = 1$. Next we compute

$$\langle \bar{v}_1, \bar{v}_1 \rangle = \int_{-1}^1 dx = 2 \tag{8}$$

$$\langle \bar{v}_1, \bar{x}_2 \rangle = \int_{-1}^1 x dx = 0 \tag{9}$$

Then,

$$\bar{v}_2 = \bar{x}_2 - \frac{\langle \bar{v}_1, \bar{x}_2 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle}\bar{v}_1 = x - \frac{0}{2}(1) = x \tag{10}$$

In order to find $\bar{v}_3$, we first compute

$$\begin{aligned}
\langle \bar{v}_1, \bar{x}_3 \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} && (11)\\
\langle \bar{v}_2, \bar{x}_3 \rangle &= \int_{-1}^{1} x^3 dx = 0 && (12)\\
\langle \bar{v}_2, \bar{v}_2 \rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3} && (13)\\
& && (14)
\end{aligned}$$

then,

$$\begin{aligned}
\bar{v}_3 &= \bar{x}_3 - \frac{\langle \bar{v}_1, \bar{x}_3 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle}\bar{v}_1 - \frac{\langle \bar{v}_2, \bar{x}_3 \rangle}{\langle \bar{v}_2, \bar{v}_2 \rangle}\bar{v}_2 && (15)\\
&= x^2 - \frac{2/3}{2}(1) - \frac{0}{2/3} && (16)\\
&= x^2 - \frac{1}{3} && (17)
\end{aligned}$$

Then, $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is an orthogonal basis for $\mathcal{P}_2$ on the interval $[-1, 1]$. The polynomials

$$1,\ x,\ x^2 - \frac{1}{3} \tag{18}$$

are the first three Legendre polynomials.

Orthogonal projection $proj_W(\bar{v})$**:** We can define orthogonal projection $\mathrm{proj}_W(\bar{v})$ of a vector $\bar{v}$ onto a subspace W of an inner product space. If $\{\bar{u}_1, \cdots, \bar{u}_k\}$ is an orthogonal basis for W, then

$$proj_W(\bar{v}) = \frac{\langle \bar{u}_1, \bar{v} \rangle}{\langle \bar{u}_1, \bar{u}_1 \rangle}\bar{u}_1 + \cdots + \frac{\langle \bar{u}_k, \bar{v} \rangle}{\langle \bar{u}_k, \bar{u}_k \rangle}\bar{u}_k \tag{19}$$

Orthogonal to W**:** The component of $\bar{v}$ orthogonal to W is the vector

$$perp_W(\bar{v}) = \bar{v} - proj_W(\bar{v}) \tag{20}$$

Remark: $proj_W(\bar{v})$ and $perp_W(\bar{v})$ are orthogonal.

The Cauchy-Schwarz and Triangle Inequalities

The Cauchy-Schwarz Inequality (Theorem 7.3): Let $\bar{u}$ and $\bar{v}$ be vectors in a inner product space V. Then

$$|\langle \bar{u}, \bar{v} \rangle| \leq ||\bar{u}||\,||\bar{v}|| \tag{21}$$

with equality holding if and only if $\bar{u}$ and $\bar{v}$ are scalar multiples of each other.

Proof: In $\bar{u} = \bar{0}$, the the inequality is actually an equality, since

$$|\langle \bar{0}, \bar{v} \rangle| = 0 = ||\bar{0}||\,||\bar{v}|| \tag{22}$$

If $\bar{u} \neq \bar{0}$, the let W be the subspace of V spanned by $\bar{u}$. Since

$$proj_W(\bar{v}) = \frac{\langle \bar{u}, \bar{v} \rangle}{\langle \bar{u}, \bar{u} \rangle}\bar{u} \tag{23}$$

and

$$perp_W(\bar{v}) = \bar{v} - proj_W(\bar{v}) \tag{24}$$

are orthogonal, we can apply Pythagoras' Theorem to obtain

$$\begin{aligned} ||\bar{v}||^2 &= ||proj_W(\bar{v}) + (\bar{v} - proj_W(\bar{v})||^2 && (25)\\ &= ||proj_W(\bar{v}) + perp_W(\bar{v})||^2 && (26)\\ &= ||proj_W(\bar{v})||^2 + ||perp_W(\bar{v})||^2 && (27) \end{aligned}$$

It follows that

$$||proj_W(\bar{v})||^2 \leq ||\bar{v}||^2 \tag{28}$$

Now

$$\begin{aligned} ||proj_W(\bar{v})||^2 &= \left\langle \frac{\langle \bar{u}, \bar{v}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u}, \frac{\langle \bar{u}, \bar{v}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u} \right\rangle && (29)\\ &= \left(\frac{\langle \bar{u}, \bar{v}\rangle}{\langle \bar{u}, \bar{u}\rangle}\right)^2 \langle \bar{u}, \bar{u}\rangle && (30)\\ &= \frac{\langle \bar{u}, \bar{v}\rangle^2}{\langle \bar{u}, \bar{u}\rangle} && (31)\\ &= \frac{\langle \bar{u}, \bar{v}\rangle^2}{||\bar{u}||^2} && (32) \end{aligned}$$

so we have

$$\frac{\langle \bar{u}, \bar{v}\rangle^2}{||\bar{u}||^2} \leq ||\bar{v}||^2 \Rightarrow \langle \bar{u}, \bar{v}\rangle^2 \leq ||\bar{v}||^2||\bar{u}||^2 \Rightarrow |\langle \bar{u}, \bar{v}\rangle| \leq ||\bar{v}||||\bar{u}|| \tag{33}$$

Clearly $|\langle \bar{u}, \bar{v}\rangle| \leq ||\bar{v}||||\bar{u}||$ is an equality if and only if $||proj_W(\bar{v})||^2 = ||\bar{v}||^2$, and this is true if and only if $perp_W(\bar{v}) = 0$, equivalently

$$\bar{v} = proj_W(\bar{v}) = \frac{\langle \bar{u}, \bar{v}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u} \tag{34}$$

If this is so, then $\bar{v}$ is a scalar multiple of $\bar{u}$. Consequently, if $\bar{v} = c\bar{u}$, then

$$\begin{aligned} perp_W(\bar{v}) &= \bar{v} - proj_W(\bar{v}) && (35)\\ &= c\bar{u} - \frac{\langle \bar{u}, c\bar{u}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u} && (36)\\ &= c\bar{u} - c\frac{\langle \bar{u}, \bar{u}\rangle}{\langle \bar{u}, \bar{u}\rangle}\bar{u} && (37)\\ &= 0 && (38) \end{aligned}$$

so equality holds in the Cauchy-Schwarz Inequality.

The Triangle Inequality (Theorem 7.4): Let $\bar{u}$ and $\bar{v}$ be vectors in an inner product space V. Then

$$||\bar{u} + \bar{v}|| \leq ||\bar{u}|| + ||\bar{v}|| \tag{39}$$

Proof: We start the demonstration with the following equality which is asked to proved in Exercise 32 in the book

$$\begin{aligned} ||\bar{u} + \bar{v}||^2 = ||\bar{u}||^2 + 2\langle \bar{u}, \bar{v}\rangle + ||\bar{v}||^2 && (40)\\ \leq ||\bar{u}||^2 + 2|\langle \bar{u}, \bar{v}\rangle| + ||\bar{v}||^2 && (41)\\ \leq ||\bar{u}||^2 + 2||\bar{u}||||\bar{v}|| + ||\bar{v}||^2 && (42)\\ \leq (||\bar{u}|| + ||\bar{v}||)^2 \Rightarrow ||\bar{u} + \bar{v}|| \leq ||\bar{u}|| + ||\bar{v}|| && (43) \end{aligned}$$

Vectors and Matrices with Complex Entries

Complex dot product: If $\bar{u}$ and $\bar{v}$ are vector in $\mathbb{C}^n$, then the complex dot product of $\bar{u}$ and $\bar{v}$ is defined by

$$\bar{u} \cdot \bar{v} = u_1^* v_1 + \cdots + u_n^* v_n \tag{44}$$

where u_i^* is the complex conjugate of u_i.

Norm: $||\bar{v}|| = \sqrt{\bar{v} \cdot \bar{v}}$

Distance: $d(\bar{u}, \bar{v}) = ||\bar{u} - \bar{v}||$

Properties:

a1. $\bar{u} \cdot \bar{v} = (\bar{v} \cdot \bar{u})^*$

a2. $\bar{u} \cdot \bar{v} = \bar{u}^+ \bar{v}$ where $\bar{u}^+$ is the conjugate transpose of $\bar{u}$.

b. $\bar{u} \cdot (\bar{v} + \bar{w}) = \bar{u} \cdot \bar{v} + \bar{u} \cdot \bar{w}$

c. $(c\bar{u}) \cdot \bar{v} = c^*(\bar{u} \cdot \bar{v})$ and $\bar{u} \cdot (c\bar{v}) = c(\bar{u} \cdot \bar{v})$

d. $\bar{u} \cdot \bar{u} \geq 0$ and $\bar{u} \cdot \bar{u} = 0$ if and only if $\bar{u} = 0$

e. For matrices with complex entries, addition, multiplication by complex scalars, transpose, and matrix multiplication are all defined exactly as we did for real matrices in Section 3.1 of the book, and the algebraic properties of these operations still hold (Section 3.2 of the book).

f. The notion of inverse and determinant of a square complex matrix are likewise the real case, and the techniques and properties all carry over the complex case (Sections 3.3. and 4.2 of the book).

However, the notion of transpose is not that useful in complex matrices. The following is an alternative useful definition when working with complex matrices:

Conjugate Transpose: If A is a complex matrix, then the conjugate transpose of A is the matrix A^+ defined by

$$A^+ = (A^T)^* \tag{45}$$

Properties of complex conjugate matrices:

a. $(A^*)^* = A$

b. $(cA)^* = c^* A^*$

c. $(A + B)^* = A^* + B^*$

d. $(AB)^* = A^* B^*$

Properties of complex transpose matrices:

a. $(A^+)^+ = A$

b. $(cA)^+ = c^* A^+$

c. $(A+B)^+ = A^+ + B^+$

d. $(AB)^+ = B^+ A^+$

The following definition is the complex generalization of real symmetric matrix.

Hermitian: A square complex matrix A is called Hermitian if $A^+ = A$–that is, if it is equal to its own conjugate transpose.

Properties of Hermitian matrices:

a. The diagonal entries of a Hermitian matrix are real

b. The eigenvalues of a Hermitian matrix are real numbers.

c. If A is Hermitian, then eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

If a square real matrix satisfies that $Q^{-1} = Q^T$ then Q is orthogonal. The next definition give the analogue for complex matrices.

Unitary: A square complex matrix U is called unitary if $U^{-1} = U^+$

Remark: In order to show that U is unitary you need only to show that $U^+ U = I$.

The following statement are equivalent:

a. U is unitary

b. The columns of U form an orthonormal set in $\mathcal{C}^n$ with respect to the complex dot product.

c. The rows of U form an orthonormal set in $\mathcal{C}^n$ with respect to the complex dot product.

d. $||U\bar{x}|| = ||\bar{x}||$ for every $\bar{x}$ in $\mathcal{C}^n$.

e. $U\bar{x} \cdot U\bar{y} = \bar{x} \cdot \bar{y}$ for every $\bar{x}$ and $\bar{y}$ in $\mathcal{C}^n$.

The following definition is the natural generalization of orthogonal diagonalizability to complex matrices

Unitary Diagonalizable: A square matrix A is called unitary diagonalizable if there exist a unitary matrix U and a diagonal matrix D such that

$$U^+AU = D \tag{46}$$

where the columns of U must form an orthonormal basis in $\mathcal{C}^n$ consisting of eigenvectors of A.

The following is the process to find U and D,

1. Compute the eigenvalues of A
2. Find a basis for each eigenspace
3. Ensure that each eigenspace basis consists of orthonormal vectors (using the Gram-Schmidt Process, with the complex dot product, if necessary)
4. Form the matrix U whose columns are the orthonormal eigenvectors just found.
5. As a consequence of this construction the product U^+AU will be a diagonal matrix D whose diagonal entries are the eigenvalues of A arranged in the order as the corresponding eigenvectors in the columns of U.

Remarks:

- Every Hermitian matrix is unitary diagonalizable. This is the Complex Spectral Theorem.
- It turns that the inverse statement, i.e. every unitary diagonalizable matrix is Hermitian, is not true.

The characterization of unitary diagonalizability is the following theorem (it is not demonstrated in the book.

Unitarily Diagonalizable: A square complex matrix A is unitarily diagonalizable if and only if

$$A^+A = AA^+ \tag{47}$$

Normal matrix: A matrix A for which $A^+A = AA^+$ is called normal.

Skew(asimétrico)-Hermitian matrix: A matrix A for which $A^+ = -A$ is called skew-Hermitian.

Remark:

- Every Hermitian matrix, every unitary matrix, and every skew-Hermitian matrix is normal. In the real case, this result refers to symmetric, orthogonal, and skew-asymmetric matrices, respectively.
- If a square complex matrix is unitarily diagonalizable, then it is normal.

Geometric Inequalities and Optimization Problems

Recall that the Cauchy-Schwarz Inequality in $\mathbb{R}^n$ states that for all vectors $\bar{u}$ and $\bar{v}$,

$$|\bar{u}\cdot\bar{v}| \leq ||\bar{u}||\,||\bar{v}|| \tag{48}$$

$$|\sum_{i=1}^{n} x_i y_i| \leq \sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} \tag{49}$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sqrt{\sum_{i=1}^{n} x_i^2}\right)^2 \left(\sqrt{\sum_{i=1}^{n} y_i^2}\right)^2 \tag{50}$$

Examples:

- $\sqrt{xy} \leq \frac{x+y}{2}$ where $\sqrt{xy}$ is the geometric mean and $(x+y)/2$ is the arithmetic mean or average.
- $(\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} x_i}{n}$

Definitions:

Quadratic Means: $\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$

Harmonic Mean: $\frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$

Relation between the different means:

$$\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2} \geq \sqrt{xy} \geq \frac{2}{\frac{1}{x}+\frac{1}{y}} \tag{51}$$

Norms and Distance Functions

Norm: A norm on a vector space V is a mapping that associates with each vector $\bar{v}$ a real number $||\bar{v}||$, called the norm of $\bar{v}$, such that the following properties are satisfied for all vector $\bar{u}$ and $\bar{v}$ and all scalars c:

1. $||\bar{v}|| \geq 0$, and $||\bar{v}|| = 0$ if and only if $\bar{v} = 0$
2. $||c\bar{v}|| = c||\bar{v}||$
3. $||\bar{u}+\bar{v}|| \leq ||\bar{u}|| + ||\bar{v}||$

Normed linear space: A vector space with a norm is called a normed linear spaced

Example:

scalar norm: An inner product space with norm $||\bar{v}|| = \sqrt{\langle \bar{v}, \bar{v}\rangle}$ defines a norm.

Sum norm or 1-norm: The sum norm $||\bar{v}||_s$ or $||\bar{v}||_1$ of a vector $\bar{v}$ in $\mathbb{R}^n$ is the sum of the absolute values of its components. That is, if $\bar{v} = [v_1, \cdots, v_n]^T$, then $||\bar{v}||_s = |v_1| + \cdots + |v_n|$ is a norm.

Max norm or ∞ norm or uniform norm: The max norm $||\bar{v}||_m$ or $||\bar{v}||_\infty$ of a vector in $\mathbb{R}^n$ is the largest number among the absolute values of its components. That is, if $\bar{v} = [v_1, \cdots, v_n]^T$, then $||\bar{v}||_m = max\{|v_1|, \cdots, |v_n|\}$ is a norm.

In general, it is possible to define a norm $||\bar{v}||_p$ of a vector in $\mathbb{R}^n$ by $||\bar{v}||_p = (|v_1|^p + \cdots + |v_n|^p)^{1/p}$ for any real $p \geq 1$. For

- $p = 1$, $||\bar{v}||_1 = ||\bar{v}||_s$.
- $p = 2$, $||\bar{v}||_2 = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$. It is the familiar norm obtain from the dot product. This 2-norm of Euclidean norm, is often denoted by $||\bar{v}||_E$.

Distance Functions

For any norm, we can define a distance function:

$$d(\bar{u}, \bar{v}) = ||\bar{u} - \bar{v}|| \tag{52}$$

Exercise for the student in class (Example 7.16): Compute the distance d$(\bar{u}, \bar{v})$ relative to

a. the Euclidean norm. You should get: 5.

b. the sum norm. You should get: 7.

c. the max norm. You should get: 4.

where $\bar{u}^T = [3, -2]$ and $\bar{v}^T = [-1, 1]$.

Properties of the distance function(Theorem 7.5): Let d be a distance function defined on a normed linear space V. The following properties hold for all vectors $\bar{u}$, $\bar{v}$, and $\bar{w}$ in V:

a. d$(\bar{u}, \bar{v}) \geq 0$, d$(\bar{u}, \bar{v})$=0 if and only if $\bar{u} = \bar{v}$

b. d$(\bar{u}, \bar{v})$=d$(\bar{v}, \bar{u})$

c. d$(\bar{u}, \bar{w}) \leq$d$(\bar{u}, \bar{v})$+d$(\bar{v}, \bar{w})$

Proof: See book, pag. 564.

Matrix Norms

A matrix norm on M_{nn} is a mapping that associates with each $n \times n$ matrix A a real number $||A||$, called the norm of A, such that the following properties are satisfied for all $n \times n$ matrices A and B and all scalars c,

1. $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = O$.
2. $||cA|| = c||A||$
3. $||A + B|| \leq ||A|| + ||B||$
4. $||AB|| \leq ||A|| \, ||B||$

Compatible: A matrix norm on M_{nn} is said to be compatible with a vector norm $||\bar{x}||$ on $\mathbb{R}^n$ if for all $n \times n$ matrices A and all vectors $\bar{x}$ in $\mathbb{R}^n$, we have

$$||A\bar{x}|| \leq ||A||\ ||\bar{x}|| \tag{53}$$

Examples:

Frobenius norm: (Example 7.18) The Frobenius norm $||A||_F$ of a matrix A is obtained by stringing(desfibrar) out the entries of the matrix and then taking the Euclidean norm,

$$||A||_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2} \tag{54}$$

Operator norm: (Theorem 7.6) If $||\bar{x}||$ is a vector norm on $\mathbb{R}^n$, then $||A|| = max_{||\bar{x}||=1}||A\bar{x}||$ defines a norm on M_{nn} that is compatible with the vector norm that induces it. The following are three examples:

Sum norm: $||A||_1 = max_{||\bar{x}||_s=1}||A\bar{x}||_s$

Euclidean norm: $||A||_2 = max_{||\bar{x}||_E=1}||A\bar{x}||_E$

Max norm: $||A||_\infty = max_{||\bar{x}||_m=1}||A\bar{x}||_m$

Theorem 7.7: Let A be an $n \times n$ matrix with columns vectors $\bar{a}_i$ and row vectors $\bar{A}_i$ for $i = 1, \cdots, n$,

a. $||A||_1 = max_{j=1,\cdots,n}||\bar{a}_j||_s = max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{ij}|$ (notar que se suman las columnas)

b. $||A||_\infty = max_{j=1,\cdots,n}||\bar{A}_j||_s = max_{i=1,\cdots,n} \sum_{j=1}^{n} |a_{ij}|$ (notar que se suman las filas)

Example 7.19: Find $||A||_1$ and $||A||_\infty$ using the Theorem 7.7 and the definition $||A|| = max_{||\bar{x}||=1}||A\bar{x}||$ for

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix} \tag{55}$$

- Using T. 7.7:

$$||A||_1 = ||\bar{a}_1||_s = |1| + |4| + |-5| = 10 \tag{56}$$
$$||A||_\infty = ||\bar{A}_4||_s = |-5| + |1| + |3| = 9 \tag{57}$$

- using the definition $||A|| = max_{||\bar{x}||=1}||A\bar{x}||$:
  (i) For $||A||_1 = max_{||\bar{x}||_{s=1}}||A\bar{x}||_s$ we see that the maximum value of 10 is achieved when we take $\bar{x} = \bar{e}_1$, then

$$||A\bar{e}_1||_s = ||\bar{a}_1||_s = 10 = ||A||_1 \tag{58}$$

  (ii) For $||A||_\infty = max_{||\bar{x}||_{m=1}}||A\bar{x}||_m$, if we take $\bar{x}^T = [-1\,1\,1]$ we obtain

$$||A\bar{x}||_m = \left\| \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\|_m \tag{59}$$
$$= \left\| \begin{bmatrix} -2 \\ -7 \\ 9 \end{bmatrix} \right\|_m = max\{|-2|, |-7|, |9|\} \tag{60}$$
$$||A\bar{x}||_m = 9 \tag{61}$$

The Condition Number of a Matrix

Ill-Conditioned matrix: A matrix A is ill-conditioned if small changes in its entries can produce large changes in the solutions to $A\bar{x} = \bar{b}$.

Well-Conditioned matrix: If small changes in the entries of a matrix A produce only small changes in the solutions to $A\bar{x} = \bar{b}$, then A is called well-conditioned.

Ill-conditioned in terms of the norm: We can use matrix norms to give a more precise way of determining when a matrix is ill-conditioned. The inequality (see book, pag. 571)

$$\frac{||\Delta\bar{x}||}{||\bar{x'}||} \leq cod(A)\frac{||\Delta A||}{||A||} \tag{62}$$

gives an upper bound on how large the relative error in the solution can be in terms of the relative error in the coefficient matrix. The larger the condition number $cond(A) = ||A^{-1}||\ ||A||$, the more ill-conditioned the matrix, since there is more "room" for the error to be large relative to the solution.

Remarks:

- The condition number
$$cond(A) = ||A^{-1}||\ ||A|| \tag{63}$$
of a matrix depends on the choice of the norm. The most commonly used norms are the operator norms $||A||_1$ and $||A||_\infty$.
- For any norm, $cond(A) \geq 1$.
- If the condition number is large relative to one compatible matrix norm, it will be large relative to any compatible matrix norm.

The Convergence of Iterative Methods

One of the most important uses of matrix norms is to establish the convergence properties of various iterative methods.

Least Squares Approximation

Best Approximation: If W is a subspace of a normed linear space V and if $\bar{v}$ is a vector in V, then the best approximation to $\bar{v}$ in W is the vector $\tilde{v}$ in W such that

$$||\bar{v} - \tilde{v}|| < ||\bar{v} - \bar{w}|| \tag{64}$$

for every vector $\bar{w}$ in W different from $\tilde{v}$.

Remark: In $\mathbb{R}^2$ or $\mathbb{R}^3$, we are used of thinking of "shorter distance" as corresponding to "perpendicular distance". In algebraic terminology, "shorter distance" relates to the notion of orthogonal projection, i.e. if W is a subspace of $\mathbb{R}^n$ and $\bar{v}$ is a vector in $\mathbb{R}^n$, then we expect $proj_W(\bar{v})$ to be the vector in W that is closest to $\bar{v}$, see Fig. 1.

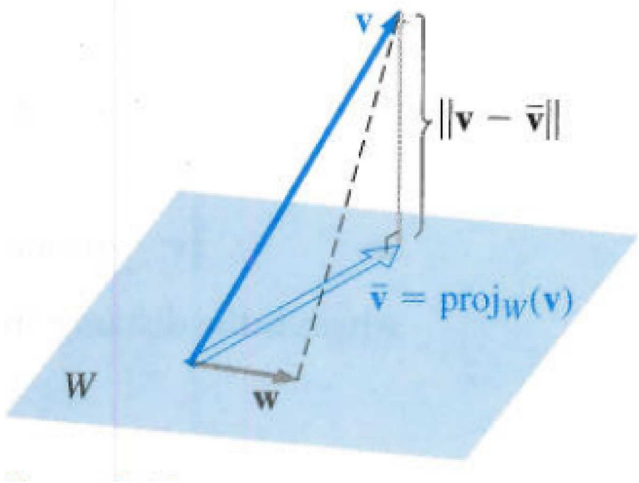

Figure 1: (from the book)

The Best Approximation Theorem (Theorem 7.8): If W is a finite-dimensional subspace of an inner product space V and if $\bar{v}$ is a vector in V, then $proj_W(\bar{v})$ is the best approximation to $\bar{v}$ in W.
Proof: Let $\bar{w}$ be a vector in W different from $proj_W(\bar{v})$. Then $proj_w(\bar{v}) - \bar{w}$ is also in W, so $\bar{v} - proj_W(\bar{v}) = perp_W(\bar{v})$ is orthogonal to $proj_W(\bar{v}) - \bar{w}$, by Exercise 43 of the book of Section 7.1. Pythagoras's Theorem now implies that

$$\begin{aligned} ||\bar{v} - proj_W(\bar{v})||^2 + ||proj_W(\bar{v}) - \bar{w}||^2 &= ||(\bar{v} - proj_W(\bar{v})) + (proj_w(\bar{v}) - \bar{w})||^2 \\ &= ||\bar{v} - \bar{w}||^2 \end{aligned} \tag{65}$$

as Fig. 1 illustrates. However, $||proj_W(\bar{v}) - \bar{w}||^2 > 0$, since $\bar{w} \neq proj_W(\bar{v})$, so

$$\begin{aligned} ||\bar{v} - proj_W(\bar{v})||^2 &< ||\bar{v} - proj_W(\bar{v})||^2 + ||proj_W(\bar{v}) - \bar{w})||^2 = ||\bar{v} - \bar{w}||^2 \\ &\Rightarrow ||\bar{v} - proj_W(\bar{v})|| < ||\bar{v} - \bar{w}|| \end{aligned} \tag{66}$$

Remark: The Best Approximation Theorem gives us an alternative proof that $proj_W(\bar{v})$ does not depend on the choice of the basis of W, since there can be only one vector in W that is closest to $\bar{v}$–namely $proj_W(\bar{v})$.

Example 7.23: Let

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \bar{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \tag{67}$$

Find the best approximation to $\bar{v}$ in the plane $W = span(\bar{u}_1, \bar{u}_2)$ and find the Euclidean distance from $\bar{v}$ to W.
Solution:
The vector in W which best approximate $\bar{v}$ is $proj_W(\bar{v})$,

$$proj_W(\bar{v}) = \left(\frac{\bar{u}_1 \cdot \bar{v}}{\bar{u}_1 \cdot \bar{u}_1}\right) \bar{u}_1 + \left(\frac{\bar{u}_2 \cdot \bar{v}}{\bar{u}_2 \cdot \bar{u}_2}\right) \bar{u}_2 \tag{68}$$

$$= \frac{2}{6} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \frac{16}{30} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2/5 \\ 1/5 \end{bmatrix} \tag{69}$$

The distance from $\bar{v}$ to W is the distance from $\bar{v}$ to the point is W closest to $\bar{v}$. But this distance is just

$$||perp_W(\bar{v})|| = ||\bar{v} - proj_w(\bar{v})|| \tag{70}$$

then

$$perp_W(\bar{v}) = \bar{v} - proj_w(\bar{v}) \tag{71}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -12/5 \\ 24/5 \end{bmatrix} \tag{72}$$

then distance from $\bar{v}$ to W is

$$||perp_W(\bar{v})|| = \sqrt{0^2 + (-12/5)^2 + (24/5)^2} = 12\sqrt{5}/5 \tag{73}$$

Least Squares Approximation

This section is about of finding a curve that "best fits" a set of data points.

Least Square Solution: If A is an $m \times n$ matrix and $\bar{b}$ is in $\mathbb{R}^m$, a least square solution of $A\bar{x} = \bar{b}$ is a vector $\tilde{x}$ in $\mathbb{R}^n$ such that

$$||\bar{b} - A\tilde{x}|| \leq ||\bar{b} - A\bar{x}|| \tag{74}$$

for all $\bar{x}$ in $\mathbb{R}^n$.

The Least Squares Theorem (Theorem 7.9): Let A be an $m \times n$ matrix and let $\bar{b}$ be in $\mathbb{R}^m$. Then $A\bar{x} = \bar{b}$ always has at least one least squares solution $\tilde{x}$. Moreover,

a. $\tilde{x}$ is a least squares solution of $A\bar{x} = \bar{b}$ if and only if $\tilde{x}$ is a solution of the normal equations $A^T A\tilde{x} = A^T\bar{b}$.

b. A has LI columns if and only if $A^T A$ is invertible. In this case, the least squares solution of $A\bar{x} = \bar{b}$ is unique and is given by $\tilde{x} = (A^T A)^{-1} A^T \bar{b}$.

Proof: See book, pag. 585.

Least Squares via the QR Factorization

If is often the case that the normal equations for a least squares problem are ill-conditioned. Therefore, a small numerical error in performing Gaussian elimination will result in a late error in the least square solution. The QR factorization of A yields a more reliable way of computing the least square approximation of $A\bar{x} = \bar{b}$.

Theorem 7.10: Let A be an $m \times n$ matrix with LI columns and let $\bar{b}$ be in $\mathbb{R}^m$. If $A = QR$ is a QR factorization of A (where Q is an $m \times n$ matrix with orthonormal columns and R is an invertible upper triangular matrix), then the unique least squares solution $\tilde{x}$ of $A\bar{x} = \bar{b}$ is

$R^{-1}Q^T\bar{b}$.
Proof: Writing $A = QR$ in $A^T A\tilde{x} = A^T\bar{b}$ we have

$$\begin{aligned} A^T A\tilde{x} &= A^T\bar{b} && (75)\\ (QR)^T QR\tilde{x} &= (QR)^T\bar{b} && (76)\\ R^T Q^T QR\tilde{x} &= R^T Q^T\bar{b} && (77)\\ R^T R\tilde{x} &= R^T Q^T\bar{b} && (78)\\ R\tilde{x} &= Q^T\bar{b} && (79)\\ \tilde{x} &= R^{-1}Q^T\bar{b} && (80)\\ & && (81) \end{aligned}$$

where we used $Q^T Q = I$ and the fact that R^T is invertible because R is so.

Remark: Since R is upper triangular, in practice it is easier to solve $R\tilde{x} = Q^T\bar{b}$.

Orthogonal Projection Revisited

The least squares method give an alternative formulation for the orthogonal projection of a vector onto a subspace on $\mathbb{R}^m$.

Theorem 7.11: Let W be a subspace of $\mathbb{R}^m$ and let A be an $m \times n$ matrix whose columns form a basis for W. If $\bar{v}$ is any vector in $\mathbb{R}^m$, then the orthogonal projection of $\bar{v}$ onto W is the vector

$$proj_W(\bar{v}) = A(A^T A)^{-1}A^T\bar{v} \tag{82}$$

The LT $P : \mathbb{R}^m \to \mathbb{R}^m$ that projects $\mathbb{R}^m$ onto W has $A(A^T A)^{-1}A^T$ as its standard matrix.
Proof: Given the way we have constructed A, its column space is W. Since the columns of A are LI, the Least Squares Theorem guarantees that there is a unique least squares solution $A\bar{x} = \bar{v}$ given by

$$\tilde{x} = (A^T A)^{-1}A^T\bar{v} \tag{83}$$

By equation

$$A\tilde{x} = proj_{col(A)}(\bar{b}) \tag{84}$$

and the above statement, we have

$$A\tilde{x} = proj_W(\bar{b}) \tag{85}$$

Therefore

$$proj_W(\bar{v}) = A((A^T A)^{-1}A^T\bar{v}) = (A(A^T A)^{-1}A^T)\bar{v} \tag{86}$$

as required.

Remark: Since the projection of a vector onto a subspace W is unique, the standard matrix of this LT (as given by Theorem 7.11) cannot depend on the choice of basis for W. That is, with a different basis for W, we have a different matrix A, but the matrix $A(A^T A)^{-1}A^T$ will be the same!!!.

The Pseudoinverse of a Matrix

If A is an $n \times n$ matrix with LI columns, then it is invertible, and the unique solution to $A\bar{x} = \bar{b}$ is $\bar{x} = A^{-1}\bar{b}$. If $m > n$ and A is $m \times n$ with LI columns, then $A\bar{x} = \bar{b}$ has no exact solution, but the best approximation is given by the unique leas squares solution $\tilde{x} = (A^T A)^{-1}A^T\bar{b}$. The matrix $(A^T A)^{-1}A^T$ therefore plays the role of an "inverse of A“ in this situation.

Seudoinverse: If A is a matrix with LI columns, then the pseudoinverse of A is the matrix A^+ defined by

$$A^+ = (A^T A)^{-1} A^T \tag{87}$$

Remarks:

- If the matrix A is $m \times n$, then the pseudoinverse A^+ is $n \times m$.
- If A is $m \times n$ matrix with LI columns, the least squares solution of $A\bar{x} = \bar{b}$ is given by $\tilde{x} = A^+\bar{b}$.
- The standard matrix of the orthogonal projection P from $\mathbb{R}^m$ onto col(A) is $[P] = AA^+$
- If A is square, then $A^+ = A^{-1}$. In this case,
  - the least square solution of $A\bar{x} = \bar{b}$ is the exact solution: $\tilde{x} = A^+\bar{b} = A^{-1}\bar{b} = \bar{x}$.
  - The projection matrix becomes $[P] = AA^+ = AA^{-1} = I$

Properties of the seudoinverse (Theorem 7.12): Let A be a matrix with LI columns. Then the pseudoinverse A^+ of A satisfies the following properties, called Penrose conditions for A:

a. $AA^+A = A$

b. $A^+AA^+ = A^+$

c. AA^+ and A^+A are symmetric.

Proof: See book, pag. 595.

The Singular Value Decomposition

Remarks:

- We saw that every symmetric matrix A can be factored as $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix displaying the eigenvalues for A.
- If A is not symmetric, such a factorization is not possible, but we may still be able to factor a square matrix A as $A = PDP^{-1}$, where D is as before but P is now simply an invertible matrix. (notar el cambio de P^T a P^{-1} para el caso de matrices no simétricas.)
- However, not every matrix is diagonalizable, but every matrix (symmetric of not, square or not) has a factorization of the form $A = PDQ^T$ (called singular value decomposition), where P and Q are orthogonal and D is a diagonal matrix.

The Singular Values of a Matrix

For any $m \times n$ matrix A, the $n \times n$ matrix A^TA is symmetric and hence can be orthogonally diagonalized, by the Spectral Theorem. Not only are the eigenvalue of A^TA all real (Theorem 5.18 of the book), they are all nonnegative: let λ be an eigenvalue of A^TA with corresponding unit eigenvector $\bar{v}$. Then

$$\begin{aligned} 0 \leq ||A\bar{v}||^2 &= (A\bar{v}) \cdot (A\bar{v}) = (A\bar{v})^T A\bar{v} = \bar{v}^T A^T A\bar{v} && (88) \\ &= \bar{v}^T \lambda \bar{v} = \lambda(\bar{v} \cdot \bar{v}) = \lambda ||\bar{v}||^2 = \lambda && (89) \end{aligned}$$

It therefore makes sense to take (positive) square roots of these eigenvalues.

Singular Values: If A is an $m \times n$ matrix, the singular values of A are the square roots of the eigenvalues of $A^T A$ and are denoted by $\sigma_1, \cdots, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \geq \cdots \geq \sigma_n$.

Remark: Consider the eigenvectors of $A^T A$ for the matrix A of $m \times n$. Since $A^T A$ is symmetric, we know that there is an orthonormal basis for $\mathbb{R}^n$ that consists of eigenvectors of $A^T A$. Let $\{\bar{v}_1, \cdots, \bar{v}_n\}$ be such a basis corresponding to the eigenvalues of $A^T A$, ordered so that $\lambda_1 \geq \cdots \geq \lambda_n$. We have

$$\lambda_i = ||A\bar{v}_i||^2 \Rightarrow \sigma_i = \sqrt{\lambda_i} = ||A\bar{v}_i|| \tag{90}$$

i.e., the singular values of A are the lengths of the vectors $A\bar{v}_1, \cdots, A\bar{v}_n$.

Geometrical interpretation: see Fig. 7.19 in the book, pag. 600.

The Singular Value Decomposition

We want to show that an $m \times n$ matrix A can be factored as

$$A = U\Sigma V^T \tag{91}$$

where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix, and Σ is and $m \times n$ "diagonal" matrix. If the nonzero singular values of A are $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$, then Σ will have the block form

$$\Sigma = \left[\begin{array}{c|c} D_{rr} & O_{r,n-r} \\ \hline O_{m-r,r} & O_{m-r,n-r} \end{array}\right] \tag{92}$$

where D is $D = diag(\sigma_i)$ with $i = 1, \cdots, r$ and O_{kl} is the zero matrix $k \times l$.

About the matrix V: To construct the orthogonal matrix V, we first find an orthonormal basis $\{\bar{v}_1, \cdots, \bar{v}_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of the $n \times n$ symmetric matrix $A^T A$. Then

$$V = [\bar{v}_1 \cdots \bar{v}_n] \tag{93}$$

is an orthogonal $n \times n$ matrix.

About the matrix U: For the orthogonal matrix U, we first note that $\{A\bar{v}_1, \cdots, A\bar{v}_n\}$ is an orthogonal set of vectors in $\mathbb{R}^m$: suppose that $\bar{v}_i$ is the eigenvector of $A^T A$ corresponding to the eigenvalue λ_i, then, for $i \neq j$,

$$(A\bar{v}_i) \cdot (A\bar{v}_j) = (A\bar{v}_i)^T A\bar{v}_j \tag{94}$$

$$= \bar{v}_i^T A^T A\bar{v}; = \bar{v}_i^T \lambda_j \bar{v}_j \tag{95}$$

$$= \lambda_j \bar{v}_i \cdot \bar{v}_j = 0 \tag{96}$$

Next we used the fact that

$$\sigma_i = ||A\bar{v}_i|| \tag{97}$$

and that the first r of these are nonzero. Therefore, we can normalize $A\bar{v}_i, \cdots, A\bar{v}_r$ by setting

$$\bar{u}_i = \frac{1}{\sigma_i} A\bar{v}_i \tag{98}$$

for $i = 1, \cdots, r$, i.e. the set $\{\bar{u}_1, \cdots, \bar{u}_r\}$ is an orthonormal set in $\mathbb{R}^m$. If it happens that $r < m$ we have to extend the set $\{\bar{u}_1, \cdots, \bar{u}_r\}$ to an orthonormal basis $\{\bar{u}_1, \cdots, \bar{u}_m\}$ form $\mathbb{R}^m$.

Then we set

$$U = [\bar{u}_1 \cdots \bar{u}_m] \tag{99}$$

Checking: It remains to be shown that this factorization works, i.e. $U\Sigma V^T = A$. Since $V^T = V^{-1}$, this is equivalent to show that

$$AV = U\Sigma \tag{100}$$

We know that $A\bar{v}_i = \sigma_i \bar{u}_i$ for $i = 1, \cdots, r$ and $||A\bar{v}_i|| = \sigma_i = 0$ for $i = r+1, \cdots, n$. Hence, $A\bar{v}_i = 0$ for $i = r+1, \cdots, n$.
Therefore,

$$\begin{aligned}
AV &= A[\bar{v}_1 \cdots \bar{v}_n] &(101)\\
&= [A\bar{v}_1 \cdots A\bar{v}_n] &(102)\\
&= [A\bar{v}_1 \cdots A\bar{v}_r \,\bar{0} \cdots \bar{0}] &(103)\\
&= [\sigma_1\bar{u}_1 \cdots \sigma_r\bar{u}_r \,\bar{0} \cdots \bar{0}] &(104)\\
&= [\bar{u}_1 \cdots \bar{u}_m] \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & O \\ 0 & \cdots & \sigma_r & \\ \hline & O & & O \end{array}\right] &(105)\\
&= U\Sigma &(106)
\end{aligned}$$

as required.

The above proved the following theorem.

The Singular Value Decomposition(SVD) (Theorem 7.13): Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \cdots = \sigma_n = 0$. Then there exist and $m \times m$ orthogonal matrix U, and $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ of the form

$$\begin{aligned}
\Sigma &= \left[\begin{array}{c|c} D_{rr} & O_{r,n-r} \\ \hline O_{m-r,r} & O_{m-r,n-r} \end{array}\right] &(107)\\
D &= \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} &(108)
\end{aligned}$$

such that

$$A = U\Sigma V^T \tag{109}$$

Left and right eigenvectors: A factorization of A as in Theorem 7.13 is called a singular value decomposition of A. The columns of U are called left singular vectors of A, and the columns of V are called right singular vectors of A. The matrices U and V are not uniquely determined by A, but Σ must contain the singular values of A.

The Outer Product Form of the SVD (Theorem 7.14): Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_n = 0$. Let $\bar{u}_1, \cdots, \bar{u}_r$ be left singular vectors and let $\bar{v}_1, \cdots, \bar{v}_r$ be right singular vectors of A corresponding to these singular values. Then

$$A = \sigma_1 \bar{u}_1 \bar{v}_1^T + \cdots + \sigma_r \bar{u}_r \bar{v}_r^T \tag{110}$$

Remark:

- The Theorem 7.13 generalizes the Spectral Theorem for positive definite, symmetric matrix.
- The Theorem 7.14 generalizes the spectral decomposition for positive definite, symmetric matrix

i.e., if A is a positive definite, symmetric matrix, then Theorems 7.13 and 7.14 reduce to the spectral theorem and decomposition respectively.

The SVD of a matrix A contains much important information about A describe in the following theorem:

Theorem 7.15 Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A. Let $\sigma_1, \cdots, \sigma_r$ be all the nonzero singular values of A. Then

a. The rank of A is r.

b. $\{\bar{u}_1, \cdots, \bar{u}_r\}$ is an orthonormal basis for col(A).

c. $\{\bar{u}_{r+1}, \cdots, \bar{u}_m\}$ is an orthonormal basis for null(A^T).

d. $\{\bar{v}_1, \cdots, \bar{v}_r\}$ is an orthonormal basis for row(A).

e. $\{\bar{v}_{r+1}, \cdots, \bar{v}_n\}$ is an orthonormal basis for null(A).

Proof: See book, pag. 606.

Theorem 7.16: Let $A = U\Sigma V^T$ be a singular value decomposition of an $m \times n$ matrix A with rank r. Then the image of the unit sphere in $\mathbb{R}^n$ under the matrix transformation that maps $\bar{x}$ to $A\bar{x}$ is

a. the surface of an ellipsoid in $\mathbb{R}^m$ if $r = n$

b. a solid ellipsoid in $\mathbb{R}^m$ if $r < n$.

Proof: See book, pag. 607.

Remark: We can describe the effect of an $m \times n$ matrix A on the unit sphere in $\mathbb{R}^n$ in terms of the effect of each factor in its SVD, $A = U\Sigma V^T$, from right to left (see Fig. 2):

1. Since V^T is an orthogonal matrix, it maps the unit sphere to itself.
2. The $m \times n$ matrix Σ does two things: (i) the diagonal entries $\sigma_{r+1} = \cdots = \sigma_n = 0$ collapse $n - r$ of the dimensions of the unit sphere, leaving an r-dimensional unit sphere, (ii) the nonzero diagonal entries $\sigma_1, \cdots, \sigma_r$ distort into an ellipsoid.
3. The orthogonal matrix U aligns the axes of this ellipsoid with the orthonormal basis vectors $\{\bar{u}_1, \cdots, \bar{u}_r\}$ in $\mathbb{R}^m$.

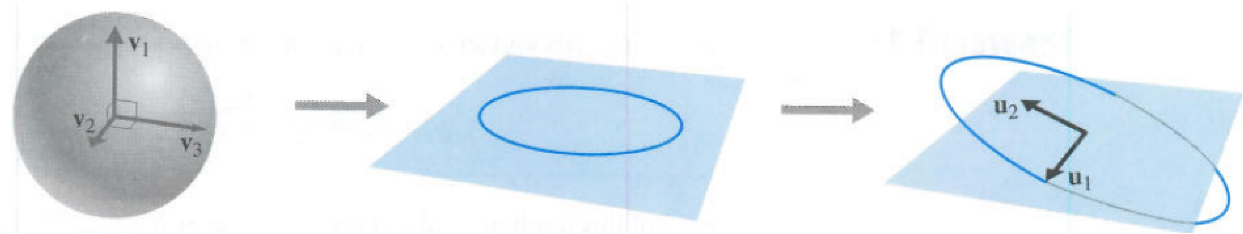

Figure 2: See details in the text (from book, Fig. 7.21, pag. 609)

Matrix Norms and the Condition Number

Theorem 7.17: Let A be an $m \times n$ matrix and let $\sigma_1, \cdots, \sigma_r$ be all the nonzero singular values of A. Then

$$||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2} \tag{111}$$

Remark:

- If A is and $m \times n$ matrix and Q is an $m \times m$ orthogonal matrix, then $||QA||_F = ||A||_F$
- $||A||_2 = max_{||\bar{x}||=1} ||A\bar{x}|| = \sigma_1$
- $cond_2(A) = |A^{-1}||_2 \, ||A||_2 = \frac{\sigma_1}{\sigma_n}$

The pseudoinverse and Least Squares Approximation

Moore-Penrose inverse (pseudoinverse): Let $A = U\Sigma V^T$ be an SVD for an $m \times n$ matrix A, where

$$\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix} \tag{112}$$

and D is an $r \times r$ diagonal matrix containing the nonzero singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ of A. The pseudoinverse or Moore-Penrose inverse of A is the $n \times m$ matrix A^+ defined by

$$A^+ = V\Sigma^+U^T \tag{113}$$

where Σ^+ is the $n \times m$ matrix

$$\Sigma^+ = \begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix} \tag{114}$$

Theorem 7.18: The least squares problem $A\bar{x} = \bar{b}$ has a unique least squares solution $\tilde{x}$ of minimal length that is given by

$$\tilde{x} = A^+\bar{b} \tag{115}$$

Remark: When A has LI columns, there is a unique least square solution $\tilde{x}$ to $A\bar{x} = \bar{b}$; that is, the normal equations $A^TA\bar{x} = A^T\bar{b}$ have the unique solution $\tilde{x} = (A^TA)^{-1}A^T\bar{b}$. When the columns of A are LD, then A^TA is not invertible, so the normal equations have infinitely many solutions. In this case, we will ask for the solution $\tilde{x}$ of minimum length. The above Theorem 7.18 fulfill this requirement.

The Fundamental Theorem of Invertible Matrices

Here we complete the Fundamental Theorem using the information that the singular value of a square matrix tell us when the matrix is invertible.

Theorem 7.19: Fundamental Theorem (FT) of Invertible Matrices. Version 5 of 5
Let A be an $n \times n$ matrix and let $T : V \to W$ be a LT whose matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively, is A. The following statements are equivalent:

From Version 1

a. A is invertible.

b. $A\bar{x} = \bar{b}$ has a unique solution for every $\bar{b}$ in $\mathbb{R}^n$.

c. $A\bar{x} = 0$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

From Version 2

f. rank(A)=n

g. nullity(A)=0

h. The column vectors of A are LI

i. The column vectors of A span $\mathbb{R}^n$

j. The column vectors of A form a basis for $\mathbb{R}^n$

k. The row vectors of A are LI

l. The row vectors of A span $\mathbb{R}^n$

m. The row vectors of A form a basis for $\mathbb{R}^n$

From Version 3

n. $det A \neq 0$

o. 0 is not an eigenvalue of A

From Version 4

p. T is invertible

q. T is one-to-one

r. T is onto

s. ker(T)=$\{\bar{0}\}$

t. range(T)=W

New statements

u. 0 is not a singular value of A

Applications

Approximation of Functions

Linear Approximation: Given a continuous function f on an interval $[a, b]$ and a subspace W of $\mathcal{C}[a, b]$, find the function "closest" to f in W. The problem is analogous to the least squares fitting of data points, except now we have infinitely many data points. The Best Approximation Theorem give the answer.

The given function f lives in the vector space $\mathcal{C}[a, b]$ of continuous functions on the interval $[a, b]$. This is an inner product space, with inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx \tag{116}$$

If W is a finite-dimensional subspace of $\mathcal{C}[a, b]$, then the best approximation to f in W is given by the projection of f onto W, by Theorem 7.8. Furthermore, if $\{\bar{u}_1, \cdots, \bar{u}_k\}$ is an orthogonal basis for W, then

$$proj_W(f) = \frac{\langle \bar{u}_1, f \rangle}{\langle \bar{u}_1, \bar{u}_1 \rangle}\bar{u}_1 + \cdots + \frac{\langle \bar{u}_k, f \rangle}{\langle \bar{u}_k, \bar{u}_k \rangle}\bar{u}_k \tag{117}$$

Example 7.41: Find the best linear approximation to $f(x) = e^x$ on the interval $[-1, 1]$.
Solution:
Linear functions are polynomials of degree 1, then we use the subspace $W = \mathcal{P}_1[-1, 1]$ of $\mathcal{C}[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. A basis for $\mathcal{P}_1[-1, 1]$ is given by $\{1, x\}$. Since

$$\langle 1, x \rangle = \int_{-1}^1 f(x)g(x)dx = 0 \tag{118}$$

this is an orthogonal basis. Then the best approximation to f in W is

$$\begin{aligned} g(x) &= proj_W(e^x) && (119)\\ &= \frac{\langle 1, e^x \rangle}{\langle 1, 1 \rangle}1 + \frac{\langle x, e^x \rangle}{\langle x, x \rangle}x && (120)\\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}x && (121)\\ &\approx 1.18 + 1.10x && (122) \end{aligned}$$

See Fig. 3.

The error is the one specified by the Best Approximation Theorem: the distance $||f - g||$ between f and g relative to the inner product we are using: (the figure 0.23 was copied from the book, pag. 621)

$$||e^x - [\frac{1}{2}(e - e^{-1}) + 3e^{-1}x]|| = \sqrt{\int_{-1}^1 \left[e^x - \frac{1}{2}(e - e^{-1}) - 3e^{-1}x\right]^2 dx} \approx 0.23 \tag{123}$$

The root mean square error can be thought of as analogous to the area between the graphs of f and g on the specified interval.

Exercise (Example 7.42): Find the best quadratic approximation to $f(x) = e^x$ on the interval $[-1, 1]$.
Solution:
A quadratic form is a polynomial of the form $g(x) = a + bx + cx^2$ in $W = \mathcal{P}_2[-1, 1]$. The standard basis $\{1, x, x^2\}$ is not orthogonal. Procedure

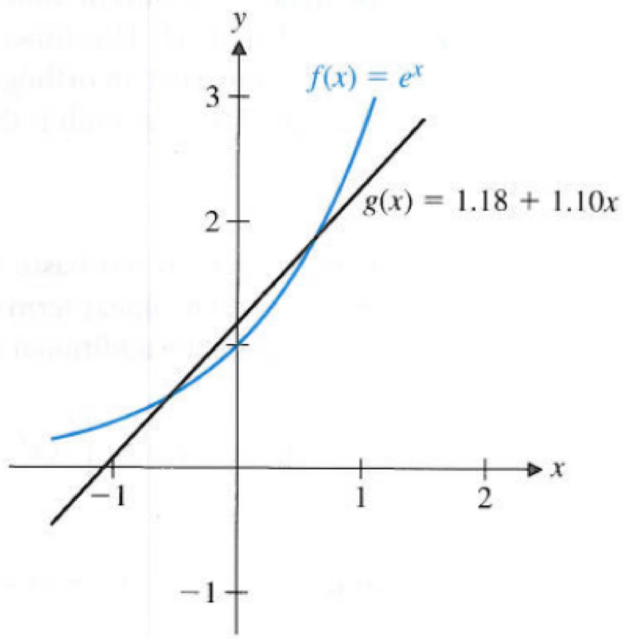

Figure 3: Best lineal approximation for e^x, see text (from the book, Fig. 7.24, pag. 621)

1. First we construct an orthogonal basis using the Gram-Schmidt Process. We should get $\{1, x, x^2 - \frac{1}{3}\}$.
2. We calculate each element on the expansion $proj_W(e^x)$. The first two element were already calculate in the previous example.

$$\langle x^2 - \frac{1}{3}, e^x\rangle = \frac{2}{3}(e - 7e^{-1}) \tag{124}$$

$$\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle = \frac{8}{45} \tag{125}$$

3. We put all term together in $proj_W(e^x)$

$$\begin{aligned} g(x) &= proj_W(e^x) && (126) \\ &= \frac{\langle 1, e^x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x, e^x\rangle}{\langle x, x\rangle}x + \frac{\langle x^2 - \frac{1}{3}, e^x\rangle}{\langle x^2 - \frac{1}{3}, x^2 - \frac{1}{3}\rangle}(x^2 - \frac{1}{3}) && (127) \\ &= \frac{1}{2}(e - e^{-1}) + 3e^{-1}x + \frac{\frac{2}{3}(e - 7e^{-1})}{\frac{8}{45}}(x^2 - \frac{1}{3}) && (128) \\ &\approx 1.00 + 1.10x + 0.54x^2 && (129) \end{aligned}$$

See Fig. 4.

The root mean square error gives $||e^x - g(x)|| \approx 0.04$.

Trigonometric Polynomial. A function of the form

$$p(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \tag{130}$$

is called a trigonometric polynomial of order n.

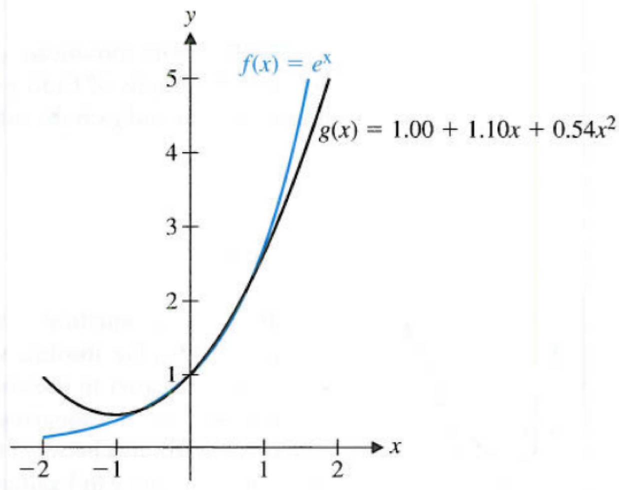

Figure 4: Best quadratic approximation of e^x, see text (from the book, Fig. 7.26, pag. 622)

Trigonometric Expansion: Let us consider the vector space $\mathcal{C}[-\pi,\pi]$ with the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \tag{131}$$

and the basis $\mathcal{B} = \{1, \cos x, \cdots, \cos nx, \sin x, \cdots, \sin nx\}$. The best approximation to a function f in $\mathcal{C}[-\pi,\pi]$ by a trigonometric polynomial of order n is $proj_W(f)$ given by

$$\begin{aligned} g(x) &= proj_W(f) \\ &= \frac{\langle 1, f\rangle}{\langle 1,1\rangle}1 + \frac{\langle \cos x, f\rangle}{\langle \cos x, \cos x\rangle}\cos x + \cdots + \frac{\langle \cos nx, f\rangle}{\langle \cos nx, \cos nx\rangle}\cos nx \\ &\quad + \frac{\langle \sin x, f\rangle}{\langle \sin x, \sin x\rangle}\sin x + \cdots + \frac{\langle \sin nx, f\rangle}{\langle \sin nx, \sin nx\rangle}\sin nx \end{aligned} \tag{132}$$

By defining the coefficients

$$a_0 = \frac{\langle 1, f\rangle}{\langle 1,1\rangle} = \frac{\langle 1, f\rangle}{2\pi} \tag{133}$$

$$a_k = \frac{\langle \cos kx, f\rangle}{\langle \cos kx, \cos kx\rangle} = \frac{\langle \cos kx, f\rangle}{\pi} \tag{134}$$

$$b_k = \frac{\langle \sin kx, f\rangle}{\langle \sin kx, \sin kx\rangle} = \frac{\langle \sin kx, f\rangle}{\pi} \tag{135}$$

where we have been used

$$\langle \cos kx, \cos kx\rangle = \int_{-\pi}^{\pi} \cos^2 kx dx = \pi \tag{136}$$

$$\langle \sin kx, \sin kx\rangle = \int_{-\pi}^{\pi} \sin^2 kx dx = \pi \tag{137}$$

$$\langle 1, 1\rangle = \int_{-\pi}^{\pi} 1^2 dx = 2\pi \tag{138}$$

Then

$$g(x) = a_0 + a_1\cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx$$

This approximation is called the nth-order Fourier approximation to f on $[-\pi, \pi]$. The coefficients $a_0, a_1, \cdots, a_n, b_1, b_n$ are called the Fourier coefficients of f and are given explicitly by the definition of the inner product,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \tag{139}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \tag{140}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \tag{141}$$