Funciones elementales

Credit: This notes are 100% from chapter 4 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

In this chapter we shall define and study a number of elementary complex analytic functions. In particular, we will investigate the complex exponential, logarithmic, power, trigonometric, hyperbolic, inverse trigonometric, and inverse hyperbolic functions. All of these functions will be shown to be analytic in a suitable domain and their derivatives will be found to agree with their real counterparts. We will also examine how these functions act as mappings of the complex plane.

Exponential and Logarithmic Functions

Complex exponential function

It is defined as

$$e^z = e^x \cos y + i e^x \sin y \tag{1}$$

Theorem (4.1): Analyticity of e^z The exponential function e^z is entire and its derivative is given by:

$$\frac{d}{dz} e^z = e^z \tag{2}$$

Modulus, Argument, and Conjugate Modulus and argument: From $w = e^z = e^x \cos y + i e^x \sin y = r(\cos\theta + i \sin\theta)$ we have

$$\begin{aligned} |e^z| &= e^x > 0 \Rightarrow e^z \neq 0 \text{ for all complex } z & (3) \\ arg(e^z) &= y + 2n\pi, \;\; n = 0, \pm 1, \pm 2, \cdots & (4) \end{aligned}$$

Conjugate:

$$\overline{(e^z)} = e^x \cos y - i e^x \sin y = e^x \cos(-y) + i e^x \sin(-y) = e^{x-iy} = e^{\bar{z}} \tag{5}$$

Theorem (4.2): Algebraic Properties If z_1 and z_2 are complex numbers, then
(i) $e^0 = 1$
(ii) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$
(iii) $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
(iv) $(e^{z_1})^n = e^{n z_1}$, $n = 0, \pm 1, \cdots$

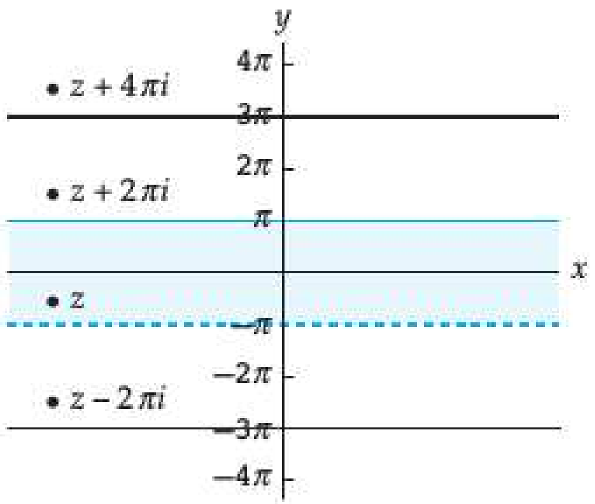

Figure 1: The fundamental region e^z (from the book)

Periodicity The most striking difference between the real and complex exponential functions is the periodicity of e^z. Analogous to real periodic functions, we say that a complex function f is periodic with period T if $f(z+T) = f(z)$ for all complex z.

The complex exponential function e^z is periodic with a pure imaginary period $2\pi i$

$$e^{z+2\pi i} = e^z \tag{6}$$
$$e^{z+2n\pi i} = e^z \tag{7}$$

for $n = 0, \pm 1, \cdots$. Thus, the complex exponential function is not one-to-one, and all values e^z are assumed in any infinite horizontal strip of width 2π in the z-plane. That is, all values of the function e^z are assumed in the set $-\infty < x < \infty$, $y_0 < y \le y_0 + 2\pi$, where y_0 is a real constant. In Figure 1 we divide the complex plane into horizontal strips obtained by setting y_0 equal to any odd multiple of π. If the point z is in the infinite horizontal strip $-\infty < x < \infty$, $-\pi < y \le \pi$, shown in color in Fig. 1, then the values $f(z) = e^z$, $f(z+2\pi i) = e^{z+2\pi i}$, $f(z-2\pi i) = e^{z-2\pi i}$, and so on are the same. The infinite horizontal strip defined by:

$$-\infty < x < \infty, \quad -\pi < y \le \pi \tag{8}$$

is called the **fundamental region** of the complex exponential function.

The Exponential Mapping

Because all values of the complex exponential function e^z are assumed in the fundamental region, the image of this region under the mapping $w = e^z$ is the same as the image of the entire complex plane.

In order to determine the image of the fundamental region under $w = e^z$, we note that this region consists of the collection of vertical line segments $z(t) = a + it$, $-\pi < t \le \pi$, where a is any real number. Then, $w(t) = e^{z(t)} = e^{a+it} = e^a e^{it}$, $-\pi < t \le \pi$, where $w(t)$ defines a circle centered at the origin with radius e^a. Because a can be any real number, the radius e^a of this circle can be any nonzero positive real number. Thus, the image of the fundamental region under the exponential mapping consists of the collection of all circles centered at the origin with nonzero radius. In other words, the image of the fundamental region $-\infty < x < \infty$, $-\pi < y \le \pi$, under $w = e^z$ is the set of all complex w with $w \ne 0$, or, equivalently, the set $|w| > 0$ (the point $w = 0$ is not in the range of the complex exponential function). See Fig. 2

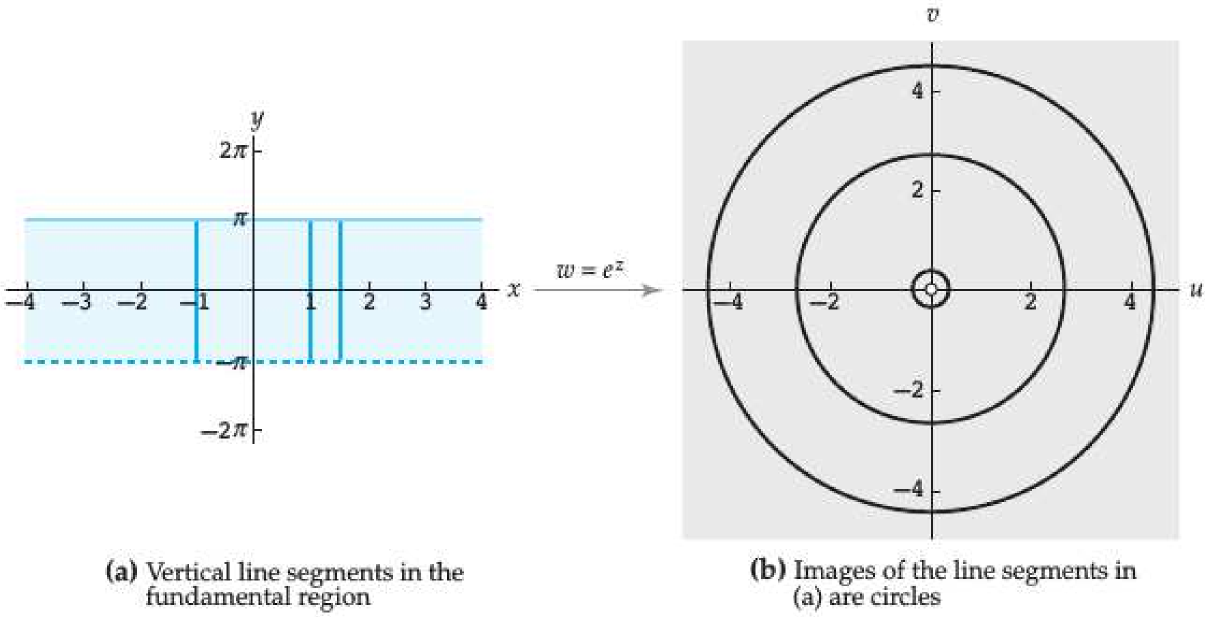

Figure 2: The image of the fundamental region under $w = e^z$ (from the book)

There was nothing particularly special about using vertical line segments to determine the image of the fundamental region under $w = e^z$. The image can also be found in the same manner by using, say, horizontal lines in the fundamental region.

Exponential Mapping Properties .
(i) $w = e^z$ maps the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, onto the set $|w| > 0$.
(ii) $w = e^z$ maps the vertical line segment $x = a$, $-\pi < y \leq \pi$, onto the circle $|w| = e^a$.
(iii) $w = e^z$ maps the horizontal line $y = b$, $-\infty < x < \infty$, onto the ray $arg(w) = b$.

Complex Logarithmic Function

In complex analysis, the complex exponential function e^z is not a one-to-one function on its domain $\mathbb{C}$. To see why the equation $e^w = z$ has infinitely many solutions, in general, suppose that $w = u + iv$ is a solution of $e^w = z$. Then we must have $|e^w| = |z|$ and $arg(e^w) = arg(z)$. It follows that $e^u = |z|$ and $v = arg(z)$, or, equivalently, $u = \ln|z|$ and $v = arg(z)$. Therefore, if

$$e^w = z \Rightarrow w = \ln|z| + iarg(z). \tag{9}$$

Because there are infinitely many arguments of z, it gives infinitely many solutions w to the equation $e^w = z$. The set of values given above defines a multiple-valued function $w = G(z)$ which is called the **complex logarithm** of z and denoted by $\ln z$, that is,

$$\ln z = \ln|z| + iarg(z) \tag{10}$$

By switching to exponential notation $z = re^{i\theta}$ we obtain the following alternative description of the complex logarithm:

$$\ln z = \ln r + i(\theta + 2n\pi) \tag{11}$$

with $n = 0, \pm 1, \pm 2, \cdots$.

The complex logarithm can be used to find all solutions to the exponential equation $e^w = z$ when z is a nonzero complex number.

Examples: Find all complex solutions to each of the following equations:

1. $e^w = i$
2. $e^w = 1 + i$
3. $e^w = -2$

Solution: For each equation $e^w = z$, the set of solutions is given by $w = \ln z$:

1. For $z = i$, we have $|z| = 1$ and $arg(z) = \pi/2 + 2n\pi$. Thus, $w = \ln i = \ln 1 + i(\pi/2 + 2n\pi) = \frac{(4n+1)\pi}{2} i$, with $n = 0, \pm 1, \pm 2, \cdots$.

2. For $z = 1 + i$, we have $|z| = \sqrt{2}$ and $arg(z) = \pi/4 + 2n\pi$. Thus, $w = \ln(1 + i) = \ln \sqrt{2} + i(\pi/4 + 2n\pi)$, with $n = 0, \pm 1, \pm 2, \cdots$.

3. For $z = -2$, we have $|z| = 2$ and $arg(z) = \pi + 2n\pi$, thus $w = \ln(-2) = \ln 2 + i(\pi + 2n\pi)$, with $n = 0, \pm 1, \pm 2, \cdots$.

Theorem (4.3): Algebraic Properties of $\ln z$ If z_1 and z_2 are nonzero complex numbers and n is an integer, then
(i) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$
(ii) $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$
(iii) $\ln z_1^n = n \ln z_1$

Principal Value of a Complex Logarithm The complex function Lnz defined by:

$$
\begin{aligned}
Lnz &= \ln|z| + iArg(z) \quad (12)\\
&= \ln r + i\theta \quad , -\pi < \theta \le \pi \quad (13)
\end{aligned}
$$

is called the principal value of the complex logarithm.

It is important to note that the identities for the complex logarithm in Theorem 4.3 are not necessarily satisfied by the principal value of the complex logarithm. For example, it is not true that $Ln(z_1 z_2) = Lnz_1 + Lnz_2$ for all complex numbers z_1 and z_2 (although it may be true for some complex numbers).

Lnz **as an Inverse Function** Because Lnz is one of the values of the complex logarithm $\ln z$, it follows for $z \ne 0$ that, $e^{Lnz} = z$. This suggests that the logarithmic function Lnz is an inverse function of exponential function e^z. Because the complex exponential function is not one-to-one on its domain, this statement is not completely accurate. The exponential function must first be restricted to the fundamental region on which it is one-to-one in order to have a well-defined inverse function, that is,

$$e^{Lnz} = z = x + iy \text{ if } -\infty < x < \infty\,, \quad -\pi < y \le \pi \quad (14)$$

Lnz **as an Inverse Function of** e^z If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \le \pi$, then f is one-to-one and the inverse function of f is the principal value of the complex logarithm $f^{-1}(z) = Lnz$.

Analyticity The principal value of the complex logarithm Lnz is discontinuous at the point $z = 0$ since this function is not defined there. This function also turns out to be discontinuous at every point on the negative real axis. This is intuitively clear since the value of Lnz a point z near the negative x-axis in the second quadrant has imaginary part close to π, whereas the value of a nearby point in the third quadrant has imaginary part close to $-\pi$. The function Lnz is, however, continuous on the set consisting of the complex plane excluding the nonpositive real axis. Therefore Lnz is a continuous function on the domain $|z| > 0$, $-\pi < arg(z) < \pi$. Put another way, the function f_1 (the **principal branch** of $\ln z$) defined by $f_1(z) = \ln r + i\theta$ is continuous on the domain $|z| > 0$, $-\pi < arg(z) < \pi$ for f_1 where $r = |z|$ and $\theta = arg(z)$. The nonpositive real axis is the **branch cut** and $z = 0$ is a **branch point**. The branch f_1 is an analytic function on its domain.

Theorem (4.4): Analyticity of the Principal Branch of $\ln z$ The principal branch f_1 of the complex logarithm defined by $f_1(z) = \ln r + i\theta$ is an analytic function and its derivative is given by:

$$f_1'(z) = \frac{1}{z} \tag{15}$$

Because $f_1(z) = Lnz$ for each point z in the domain $|z| > 0$, $-\pi < arg(z) < \pi$, it follows from Theorem 4.4 that Lnz is differentiable in this domain, and that its derivative is given by f_1'. That is, $|z| > 0$, $-\pi < arg(z) < \pi$ then:

$$\frac{d}{dz}Ln(z) = \frac{1}{z} \tag{16}$$

Logarithmic Mapping The complex logarithmic mapping $w = Lnz$ can be understood in terms of the exponential mapping $w = e^z$ since these functions are inverses of each other. The following summarizes some of these properties.
(i) $w = Lnz$ maps the set $|z| > 0$ onto the region $-\infty < u < \infty$, $-\pi < v \leq \pi$.
(ii) $w = Lnz$ maps the circle $|z| = r$ onto the vertical line segment $u = \ln r$, $-\pi < v \leq \pi$.
(iii) $w = Lnz$ maps the ray $arg(z) = \theta$ onto the horizontal line $v = \theta$, $-\infty < u < \infty$.

Complex Powers

If α is a complex number and $z \neq 0$, then the complex power z^α is defined to be:

$$z^\alpha = e^{\alpha \ln z} \tag{17}$$

In general, $e^{\alpha \ln z}$ gives an infinite set of values because the complex logarithm $\ln z$ is multiple-valued. When n is an integer, however, the expression it is single-valued (in agreement with fact that z^n is a function when n is an integer).

Examples: Find the values of the given complex power:
(a) i^{2i}
(b) $(1+i)^i$
Solution:
(a) Since $\ln i = \frac{(4n+1)\pi}{2}i$ then

$$i^{2i} = e^{2i \ln i} = e^{-(4n+1)\pi} \tag{18}$$

(b) Since $\ln(1+i) = \frac{1}{2}\ln 2 + \frac{(8n+1)\pi}{4}i$ then

$$(1+i)^i = e^{i\ln(1+i)} = e^{-(8n+1)\pi/4 + i(\ln 2)/2} \tag{19}$$

Properties: .
(i) $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1+\alpha_2}$
(ii) $\frac{z^{\alpha_1}}{z^{\alpha_2}} = z^{\alpha_1-\alpha_2}$
(iii) $(z^\alpha)^n = z^{n\alpha}$

Principal Value of a Complex Power If α is a complex number and $z \neq 0$, then the function defined by:

$$z^\alpha = e^{\alpha Lnz} \tag{20}$$

is called the principal value of the complex power z^α.

Analyticity In general, the principal value of a complex power $z^\alpha = e^{\alpha Lnz}$ is not a continuous function on the complex plane because the function Lnz is not continuous on the complex plane. However, since the function $e^{\alpha z}$ is continuous on the entire complex plane, and since the function Lnz is continuous on the domain $|z| > 0$, $-\pi < arg(z) < \pi$, it follows that z^α is continuous on the domain $|z| > 0$, $-\pi < arg(z) < \pi$. Using polar coordinates $r = |z|$ and $\theta = arg(z)$ we have found that the function defined by:

$$f_1(z) = e^{\alpha(\ln r + i\theta)} \tag{21}$$

$-\pi < \theta < \pi$, is a branch of the multiple-valued function $F(z) = z^\alpha = e^{\alpha \ln z}$. This particular branch is called the **principal branch of the complex power** z^α; its branch cut is the nonpositive real axis, and $z = 0$ is a branch point.

The branch f_1 agrees with the principal value z^α on the domain $|z| > 0$, $-\pi < arg(z) < \pi$. Consequently, the derivative of f_1 can be found using the chain rule:

$$\begin{aligned} f_1'(z) &= \frac{d}{dz} e^{\alpha Lnz} = e^{\alpha Lnz} \frac{d}{dz}(\alpha Lnz) = e^{\alpha Lnz} \frac{\alpha}{z} = z^\alpha \frac{\alpha}{z} = \alpha z^{\alpha-1} \\ \frac{d}{dz} e^{z^\alpha} &= \alpha z^{\alpha-1} \end{aligned} \tag{22}$$

Remarks: .
(i) $(z^{\alpha_1})^{\alpha_2} \neq z^{\alpha_1 \alpha_2}$ unless α_2 is an integer.
(ii) Some properties that do hold for complex powers do not hold for principal values of complex powers. For example, we can prove that $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$ for any nonzero complex numbers z_1 and z_2. However, this property does not hold for principal values of these complex powers.

Trigonometric and Hyperbolic Functions

Complex Sine and Cosine Functions The complex sine and cosine functions are defined by:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{23}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{24}$$

Analogous to real trigonometric functions, we next define the **complex tangent**, **cotangent**, **secant**, and **cosecant** functions using the complex sine and cosine:

$$\tan z = \frac{\sin z}{\cos z} \qquad \cot z = \frac{\cos z}{\sin z} \tag{25}$$

$$\sec z = \frac{1}{\cos z} \qquad \csc z = \frac{1}{\sin z} \tag{26}$$

Trigonometric identities

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z \tag{27}$$
$$\cos^2 z + \sin^2 z = 1 \tag{28}$$
$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \tag{29}$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \tag{30}$$

Periodicity The complex sine and cosine are periodic functions with a real period of 2π

$$\sin(z + 2\pi) = \sin z \tag{31}$$
$$\cos(z + 2\pi) = \cos z \tag{32}$$

Example: Find all solutions to the equation $\sin z = 5$.
Solution: from

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 5 \tag{33}$$

we build a quadratic equation for e^{iz}

$$e^{2iz} - 10ie^{iz} - 1 = 0 \tag{34}$$

which gives

$$e^{iz} = (5 \pm 2\sqrt{6})i \tag{35}$$

then

$$z = -i\ln(5 + 2\sqrt{6})i = \frac{(4n+1)\pi}{2} - i\ln(5 + 2\sqrt{6}) \tag{36}$$
$$z = -i\ln(5 - 2\sqrt{6})i = \frac{(4n+1)\pi}{2} - i\ln(5 - 2\sqrt{6}) \tag{37}$$

Modulus The modulus of a complex trigonometric function can also be helpful in solving trigonometric equations. To find a formula in terms of x and y for the modulus of the sine and cosine functions, we first express these functions in terms of their real and imaginary parts.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \tag{38}$$
$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \tag{39}$$
$$= \sin x\frac{e^{y} + e^{-y}}{2} + i\cos x\frac{e^{y} - e^{-y}}{2} \tag{40}$$
$$\sin z = \sin x\cosh y + i\cos x\sinh y \tag{41}$$

and

$$\cos z = \cos x\cosh y - i\sin x\sinh y \tag{42}$$

Then,

$$|\sin z| = \sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y} \tag{43}$$
$$= \sqrt{\sin^2 x + \sinh^2 y} \tag{44}$$
$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y} \tag{45}$$

where we have used $\sin^2 + \cos^2 = 1$.

Zeros It is a natural question to ask whether the complex sine and cosine functions have any additional zeros in the complex plane. One way to find the zero is by recognizing that a complex number is equal to 0 if and only if its modulus is 0. Thus, solving the equation $\sin z = 0$ is equivalent to solving the equation $|\sin z| = 0$, then

$$|\sin z|^2 = \sin^2 x + \sinh^2 y = 0 \tag{46}$$

Since $\sin^2 x$ and $\sinh^2 y$ are both nonnegative real numbers, this equation is satisfied if and only if $\sin x = 0$ and $\sinh y = 0$, that is, when $x = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$, $y = 0$. That is, the zeros of the complex sine function are the same as the zeros of the real sine functions.

The only zeros of the complex cosine function are the real numbers $z = (2n + 1)\pi/2$, $n = 0, \pm 1, \cdots$.

Analyticity The derivatives of the complex sine and cosine functions are found using the chain rule, we get

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z \tag{47}$$

$$\frac{d}{dz}\cos z = -\sin z \tag{48}$$

Since this derivative is defined for all complex z, $\sin z$ and $\cos z$ are an entire functions.

The derivatives of $\sin z$ and $\cos z$ can then be used to show that derivatives of all of the complex trigonometric functions are the same as derivatives of the real trigonometric functions:

$$\frac{d}{dz}\sin z = \cos z \tag{49}$$

$$\frac{d}{dz}\cos z = -\sin z \tag{50}$$

$$\frac{d}{dz}\tan z = \sec^2 z \tag{51}$$

$$\frac{d}{dz}\cot z = -\csc^2 z \tag{52}$$

$$\frac{d}{dz}\sec z = \sec z \tan z \tag{53}$$

$$\frac{d}{dz}\csc z = -\csc z \cot z \tag{54}$$

The sine and cosine functions are entire, but the tangent and secant functions have singularities at $z = (2n + 1)\pi/2$ for $n = 0, \pm 1, \cdots$, whereas the cotangent and cosecant functions have singularities at $z = n\pi$ for $n = 0, \pm 1, \cdots$.

Complex Hyperbolic Functions

Complex Hyperbolic Sine and Cosine The complex hyperbolic sine and hyperbolic cosine functions are defined by:

$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{55}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{56}$$

The complex hyperbolic functions are periodic and have infinitely many zeros.

The complex hyperbolic tangent, cotangent, secant, and cosecant are defined in terms of $\sinh z$ and $\cosh z$:

$$\tanh z = \frac{\sinh z}{\cosh z} \quad \coth z = \frac{\cosh z}{\sinh z} \tag{57}$$

$$sechz = \frac{1}{\cosh z} \quad cschz = \frac{1}{\sinh z} \tag{58}$$

The hyperbolic sine and cosine functions are entire because the functions e^z and e^{-z} are entire.

From the chain rule we have:

$$\frac{d}{dz}\sinh z = \cosh z \tag{59}$$

$$\frac{d}{dz}\cosh z = \sinh z \tag{60}$$

$$\frac{d}{dz}\tanh z = sech^2 z \tag{61}$$

$$\frac{d}{dz}\coth z = -csch^2 z \tag{62}$$

$$\frac{d}{dz}sechz = -sechz \tanh z \tag{63}$$

$$\frac{d}{dz}cschz = -cschz \coth z \tag{64}$$

Relation to sine and cosine By replacing z with iz in the definition of $\sinh z$ we have

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i \sin z \Rightarrow \sin(z) = -i \sinh(iz) \tag{65}$$

In a similar manner can be obtained others identities,

$$\sin(z) = -i \sinh(iz) \tag{66}$$

$$\cos(z) = \cosh(iz) \tag{67}$$

$$\sinh(z) = -i \sin(iz) \tag{68}$$

$$\cosh(z) = \cos(iz) \tag{69}$$

Relations between the other trigonometric and hyperbolic functions can now be derived from the above ones, for example,

$$\tan(iz) = i \tanh(z) \tag{70}$$

We can also use above relations to derive hyperbolic identities from trigonometric identities,

$$\sinh(-z) = -\sinh z \quad \cosh(-z) = \cosh z \tag{71}$$

$$\cosh^2 z - \sinh^2 z = 1 \tag{72}$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2 \tag{73}$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2 \tag{74}$$

Inverse Trigonometric and Hyperbolic Functions

The complex sine function is periodic with a real period of 2π. We also known that the sine function maps the complex plane onto the complex plane. These two properties imply that for any complex number z there exists infinitely many solutions w to the equation $\sin w = z$. Let us find an explicit formula for w

$$\sin w = z = \frac{e^{iw} - e^{-iw}}{2i} \Rightarrow e^{i2w} - 2ize^{iw} - 1 = 0 \tag{75}$$

then

$$e^{iw} = iz + (1-z^2)^{1/2} \tag{76}$$

where $(1-z^2)^{1/2}$ represents the *two* square roots of $1-z^2$.

Finally, we solve for w using the complex logarithm:

$$w = -i \ln \left[iz + (1-z^2)^{1/2}\right] \tag{77}$$

Each value of w obtained from the above equation satisfies the equation $\sin w = z$. Therefore, we call this multiple-valued function the inverse sine:

Inverse Sine The multiple-valued function $\sin^{-1} z$ defined by:

$$\sin^{-1} z = arcsinz = -i \ln \left[iz + (1-z^2)^{1/2}\right] \tag{78}$$

is called the inverse sine. The inverse sine is multiple-valued since it is defined in terms of the complex logarithm $\ln z$. It is also worth repeating that the expression $(1-z^2)^{1/2}$ represents the two square roots of $1-z^2$.

Example Find all values of $sin^{-1}\sqrt{5}$.
Solution: By setting $z = \sqrt{5}$ we get,

$$\sin^{-1}\sqrt{5} = -i \ln \left[i\sqrt{5} + \left(1 - (\sqrt{5})^2\right)^{1/2}\right] \tag{79}$$

$$= -i \ln \left[i\sqrt{5} + (-4)^{1/2}\right] \tag{80}$$

The two square roots $(-4)^{1/2}$ of –4 are found to be $\pm 2i$, then

$$\sin^{-1}\sqrt{5} = -i \ln \left[i\sqrt{5} \pm 2i\right] = -i \ln \left[i\left(\sqrt{5} \pm 2\right)\right] \tag{81}$$

Besides,

$$\ln \left[i\left(\sqrt{5} \pm 2\right)\right] = \ln |\left(\sqrt{5} \pm 2\right)| + i\left(Arg\left[i\left(\sqrt{5} \pm 2\right)\right] + 2n\pi\right) \tag{82}$$

$$= \ln\left(\sqrt{5} \pm 2\right) + i\left(\frac{\pi}{2} + 2n\pi\right) \tag{83}$$

Let us noticing the following identity,

$$\ln\left(\sqrt{5} - 2\right) = \ln\left[(\sqrt{5} - 2)\frac{\sqrt{5}+2}{\sqrt{5}+2}\right] \tag{84}$$

$$= \ln\left[\frac{5-4}{\sqrt{5}+2}\right] \tag{85}$$

$$= \ln\left[\frac{1}{\sqrt{5}+2}\right] \tag{86}$$

$$= -\ln\left(\sqrt{5}+2\right) \tag{87}$$

which implies,

$$\ln\left[i\left(\sqrt{5}\pm 2\right)\right] = \pm\ln\left(\sqrt{5}+2\right)+i\left(\frac{\pi}{2}+2n\pi\right) \tag{88}$$

Then,

$$\begin{aligned}
\sin^{-1}\sqrt{5} &= -i\ln\left[i\left(\sqrt{5}\pm 2\right)\right] &(89)\\
&= (-i)\left[\pm\ln\left(\sqrt{5}+2\right)+i\left(\frac{\pi}{2}+2n\pi\right)\right] &(90)\\
&= \mp i\ln\left(\sqrt{5}+2\right)+\left(\frac{\pi}{2}+2n\pi\right) &(91)\\
&= \frac{1+4n}{2}\pi \mp i\ln\left(\sqrt{5}+2\right) &(92)
\end{aligned}$$

Inverse cosine and inverse tangent

$$\begin{aligned}
\cos^{-1}z &= -i\ln\left[z+i(1-z^2)^{1/2}\right] &(93)\\
\tan^{-1}z &= \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) &(94)
\end{aligned}$$

Both the inverse cosine and inverse tangent are multiple-valued functions since they are defined in terms of the complex logarithm $\ln z$.

Branches and Analyticity The inverse sine and inverse cosine are multiple-valued functions that can be made single-valued by specifying a single value of the square root to use for the expression $(1-z^2)^{1/2}$ and a single value of the complex logarithm to use. The inverse tangent, on the other hand, can be made single-valued by just specifying a single value of $\ln z$ to use.

Example: We can define a function f that gives a value of the inverse sine by using the principal square root and the principal value of the complex logarithm. If, say, $z=\sqrt{5}$, then the principal square root of $1-(\sqrt{5})^2=-4$ is $2i$, and $Ln(i\sqrt{5}+2i)=\ln(\sqrt{5}+2)+i\pi/2$, then

$$f(\sqrt{5})=\frac{\pi}{2}-i\ln(\sqrt{5}+2) \tag{95}$$

Thus, we see that the value of the function f at $z=\sqrt{5}$ is the value of $sin^{-1}\sqrt{5}$ associated to $n=0$ and the square root $2i$ in the example above.

A branch of a multiple-valued inverse trigonometric function may be obtained by choosing a branch of the square root function and a branch of the complex logarithm. Determining the domain of a branch defined in this manner can be quite involved.

Derivatives of Branches $\sin^{-1}z$, $\cos^{-1}z$, and $\tan^{-1}z$ The following formulas for the derivatives hold only on the domains of these branches,

$$\begin{aligned}
\frac{d}{dz}\sin^{-1}z &= \frac{1}{(1-z^2)^{1/2}} &(96)\\
\frac{d}{dz}\cos^{-1}z &= \frac{-1}{(1-z^2)^{1/2}} &(97)\\
\frac{d}{dz}\tan^{-1}z &= \frac{1}{1+z^2} &(98)
\end{aligned}$$

Inverse Hyperbolic Functions The foregoing discussion of inverse trigonometric functions can be repeated for hyperbolic functions. This leads to the definition of the inverse hyperbolic functions stated below. Once again these inverses are defined in terms of the complex logarithm because the hyperbolic functions are defined in terms of the complex exponential.

Inverse Hyperbolic Sine, Cosine, and Tangent The multiple-valued functions $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$, defined by:

$$\sinh^{-1} z = \ln\left[z + (z^2+1)^{1/2}\right] \tag{99}$$

$$\cosh^{-1} z = \ln\left[z + (z^2-1)^{1/2}\right] \tag{100}$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \tag{101}$$

These expressions allow us to solve equations involving the complex hyperbolic functions. In particular, if $w = \sinh^{-1} z$, then $\sinh w = z$.

Branches of the inverse hyperbolic functions are defined by choosing branches of the square root and complex logarithm, or, in the case of the inverse hyperbolic tangent, just choosing a branch of the complex logarithm. The derivative of a branch can be found using implicit differentiation. The following result gives formulas for the derivatives of branches of the inverse hyperbolic functions. In these formulas, the symbols $\sinh^{-1} z$, $\cosh^{-1} z$, and $\tanh^{-1} z$ represent branches of the corresponding inverse hyperbolic multiple-valued functions.

$$\frac{d}{dz}\sinh^{-1} z = \frac{1}{(z^2+1)^{1/2}} \tag{102}$$

$$\frac{d}{dz}\cosh^{-1} z = \frac{1}{(z^2-1)^{1/2}} \tag{103}$$

$$\frac{d}{dz}\tanh^{-1} z = \frac{-1}{z^2-1} \tag{104}$$