Funciones analíticas

Credit: This notes are 100% from chapter 3 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

Differentiability and Analyticity

Derivative of Complex Function Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0, denoted by $f'(z_0)$, is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1}$$

provided this limit exists.

If the above limit exists, then the function f is said to be **differentiable** at z_0.

Rules of Differentiation If f and g are differentiable at a point z, and c is a complex constant, then

Constant Rules: $\frac{d}{dz}c = 0$ and $\frac{d}{dz}cf(z) = cf'(z)$

Sum Rule: $\frac{d}{dz}[f(z) \pm g(z)] = f'(z) \pm g'(z)$

Product Rule: $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Quotient Rule: $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z)-f(z)g'(z)}{[g(z)]^2}$

Chain Rule: $\frac{d}{dz}f(g((z)) = f'(g(z))g'(z)$

Power Rule: $\frac{d}{dz}z^n = nz^{n-1}$

Power Rule for functions: $\frac{d}{dz}[g(z)]^n = n[g(z)]^{n-1}g'(z)$

The basic 'Power Rule' does not apply to powers of the conjugate of z because the function $f(z) = \bar{z}$ is nowhere differentiable.

Analytic Functions

Analyticity at a Point A complex function $w = f(z)$ is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

Analyticity at a point is not the same as differentiability at a point. Analyticity at a point is a neighborhood property; in other words, analyticity is a property that is defined over an open set.

Analyticity in a domain A function f is analytic in a domain D if it is analytic at every point in D.

Holomorphic or regular function A function f that is analytic throughout a domain D is called holomorphic or regular.

Entire Functions A function that is analytic at every point z in the complex plane is said to be an entire function. In view of differentiation we can conclude that polynomial functions are differentiable at every point z in the complex plane and rational functions are analytic throughout any domain D that contains no points at which the denominator is zero. The following theorem summarizes these results.

Theorem (3.1): Polynomial and Rational Functions

i. A polynomial function $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a nonnegative integer, is an entire function.

ii. A rational function $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomial functions, is analytic in any domain D that contains no point z_0 for which $q(z_0) = 0$.

Singular Points A point z at which a complex function $w = f(z)$ fails to be analytic is called a **singular point** of f. For example, the rational function $f(z) = 4z/(z^2 - 2z + 2)$ is discontinuous at $1 + i$ and $1 - i$, f fails to be analytic at these points. Thus by (ii) of Theorem 3.1, f is not analytic in any domain containing one or both of these points.

Analyticity of Sum, Product, and Quotient If the functions f and g are analytic in a domain D then: the sum $f(z) + g(z)$, difference $f(z) - g(z)$, and product $f(z)g(z)$ are analytic. The quotient $f(z)/g(z)$ is analytic provided $g(z) \neq 0$ in D.

An Alternative Definition of $f'(z)$ Sometimes it is convenient to define the derivative of a function f using

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

As in real analysis, if a function f is differentiable at a point, the function is necessarily continuous at the point.

Theorem (3.2): Differentiability Implies Continuity If f is differentiable at a point z_0 in a domain D, then f is continuous at z_0.

Of course the converse of Theorem 3.2 is not true; continuity of a function f at a point does not guarantee that f is differentiable at the point. It follows from Theorem 2.3 that the simple function $f(z) = x + 4iy$ is continuous everywhere because the real and imaginary parts of f , $u(x, y) = x$ and $v(x, y) = 4y$ are continuous at any point (x, y). Yet we saw in Example 3 in the book that $f(z) = x + 4iy$ is not differentiable at any point z.

As another consequence of differentiability, L'Hôopital's rule for computing limits of the indeterminate form $0/0$, carries over to complex analysis.

Theorem (3.3) L'Hôpital's Rule Suppose f and g are functions that are analytic at a point z_0 and $f(z_0) = 0$, $g(z_0) = 0$, but $g'(z_0) \neq 0$. Then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \tag{3}$$

Cauchy-Riemann Equations

A function f is analytic in a domain D if f is differentiable at all points in D. We shall now develop a test for analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$ that is based on partial derivatives of its real and imaginary parts u and v.

Theorem (3.4): Cauchy-Riemann Equations Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{4}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5}$$

Proof: see pages 152 and 153 of the book.

If the Cauchy-Riemann equations are not satisfied at a point z, then f cannot be differentiable at z. We have already seen in Example 3 of Section 3.1 of book that $f(z) = x + 4iy$ is not differentiable at any point z. If we identify $u = x$ and $v = 4y$, then $\partial u/\partial x = 1$, $\partial v/\partial y = 4$, $\partial u/\partial y = 0$, and $\partial v/\partial x = 0$. In view of $\partial u/\partial x = 1 \neq \partial v/\partial y = 4$, f is nowhere differentiable.

Criterion for Non-analyticity If the Cauchy-Riemann equations are not satisfied at every point z in a domain D, then the function $f(z) = u(x, y) + iv(x, y)$ cannot be analytic in D.

Theorem (3.5): Criterion for Analyticity Suppose the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Recall that analyticity implies differentiability but not conversely. Theorem 3.5 has an analogue that gives the following criterion for differentiability.

Sufficient Conditions for Differentiability If the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point z, and if u and v satisfy the Cauchy-Riemann equations at z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f(z)$ is given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{6}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{7}$$

The following theorem is a direct consequence of the Cauchy-Riemann equations.

Theorem (3.6): Constant Functions Suppose the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D.
(i) If $|f(z)|$ is constant in D, then so is $f(z)$.
(ii) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a constant.

Polar Coordinates

For $f(z) = u(r,\theta) + iv(r,\theta)$ the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{8}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \tag{9}$$

The derivative of the function $f(z)$ in polar coordinates reads,

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \tag{10}$$

$$= \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) \tag{11}$$

About the exponential function $f(z) = e^z$ is differentiable everywhere and $f'(z) = f(z)$.

Harmonic Functions

In Section 5.5 of the book you shall see that when a complex function $f(z) = u(x,y)+iv(x,y)$ is analytic at a point z, then all the derivatives of f : $f'(z)$, $f''(z)$, $f'''(z)$, $\cdots$ are also analytic at z. As a consequence of this remarkable fact, we can conclude that all partial derivatives of the real functions $u(x,y)$ and $v(x,y)$ are continuous at z. From the continuity of the partial derivatives we then know that the second-order mixed partial derivatives are equal. This last fact, coupled with the Cauchy-Riemann equations, will be used in this section to demonstrate that there is a connection between the real and imaginary parts of an analytic function $f(z) = u(x,y)+iv(x,y)$ and the second-order partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{12}$$

known as Laplace's equation.

Harmonic Functions A real-valued function ϕ of two real variables x and y that has continuous first and second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

Theorem (3.7): Harmonic Functions Suppose the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. Then the functions $u(x,y)$ and $v(x,y)$ are harmonic in D, i.e.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{13}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{14}$$

Harmonic Conjugate Functions We have just shown that if a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its real and imaginary parts u and v are necessarily harmonic in D. Now suppose $u(x,y)$ is a given real function that is known to be harmonic in D. If it is possible to find another real harmonic function $v(x,y)$ so that u and v satisfy the Cauchy-Riemann equations throughout the domain D, then the function $v(x, y)$ is called a **harmonic conjugate** of $u(x, y)$. By combining the functions as $u(x, y) + iv(x, y)$ we obtain a function that is analytic in D.

About the invariance of Laplace's equation under mapping There is another important connection between analytic functions and Laplace's equation. In applied mathematics it is often the case that we wish to solve Laplace's equation $\nabla^2\phi$ in a domain D in the xy-plane, and for reasons that depend in a very fundamental manner on the shape of D, it simply may not be possible to determine ϕ. But it may be possible to devise a special analytic mapping $f(z) = u(x, y) + iv(x, y)$ or $u = u(x, y)$, $v = v(x, y)$, from the xy-plane to the uv-plane so that D', the image of D, not only has a more convenient shape but the function $\phi(x, y)$ that satisfies Laplace's equation in D also satisfies Laplace's equation in D'. We then solve Laplace's equation in D' (the solution Φ will be a function of u and v) and then return to the xy-plane and $\phi(x, y)$.