Funciones complejas y mapeos

Credit: This notes are 100% from chapter 2 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

In this chapter it is studied the functions from a set of complex numbers to another set of complex numbers.

Complex Functions

A complex function or a complex-valued function of a complex variable is a function f whose domain and range are subsets of the set $\mathbb{C}$ of complex numbers.

Real and Imaginary Parts of a Complex Function: If $w = f(z)$ is a complex function, then the image of a complex number $z = x + iy$ under f is a complex number $w = u + iv = f(x + iy) = u(x, y) + i\,v(x, y)$. The functions $u(x, y)$ and $v(x, y)$ are called the **real** and **imaginary parts of** f, respectively. **Example:** $f(z) = z^2 - (2+i)z \Rightarrow u(x, y) = x^2 - 2x + y - y^2$ and $v(x, y) = 2xy - x - 2y$.

Comments:

- Every complex function is completely determined by the real functions $u(x, y)$ and $v(x, y)$.
- Complex functions defined in terms of $u(x, y)$ and $v(x, y)$ can always be expressed, if desired, in terms of operations on the symbols z and $\bar{z}$.

Complex exponential function: This complex function is an example of one that is defined by specifying its real and Imaginary parts, $e^z = e^x \cos y + ie^x \sin y$.

Exponential form of a complex number: the polar form of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ can be written as $z = re^{i\theta}$, called the exponential form of the complex number z.

Properties:

- Notice that the exponential form is not unique since $\theta = arg(z)$ is not unique.
- $e^0 = 1$
- $e^{z_1}e^{z_2} = e^{z_1+z_2}$
- $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

- $(e^{z_1})^n = e^{nz_1}$ for $n = 0, \pm 1, \pm 2, \cdots$
- The complex exponential function is periodic: $e^{z+2\pi i} = e^z$ for all complex numbers z.

Polar coordinates: Using the polar form of $z = re^{i\theta}$ we can write the function $f(z)$ in its real and imaginary parts, $f(z) = u(r, \theta) + iv(r, \theta)$. A complex function can be defined by specifying its real and imaginary parts in polar coordinates.

Complex Functions as Mappings

The graph $(z, f(z))$ of a complex function lies in four-dimensional space then, we cannot use graphs to study complex functions. Every complex function describes a correspondence between points in two copies of the complex plane. Specifically, the point z in the z-plane is associated with the unique point $w = f(z)$ in the w-plane. We use the alternative term **complex mapping** in place of "complex function" when considering the function as this correspondence between points in the z-plane and points in the w-plane. The geometric representation of a complex mapping $w = f(z)$ consists of two figures: the first, a subset S of points in the z-plane, and the second, the set S' of the images of points in S under $w = f(z)$ in the w-plane. Then, if the point z_0 in the z-plane corresponds to the point w_0 in the w-plane, that is, if $w_0 = f(z_0)$, then we say that f **maps** z_0 onto w_0 or, equivalently, that z_0 is **mapped** onto w_0 by f.

Image of S under f: If $w = f(z)$ is a complex mapping and if S (**pre-image of** S') is a set of points in the z-plane, then we call the set of images of the points in S under f the **image of** S **under** f, and we denoted this set by the symbol S'.

Exercise (Example 2): Find the image of the line $x = 1$ under the complex mapping $w = z^2$ and represent the mapping graphically.
Solution:
Let C be the set of points on the vertical line $x = 1$, i.e $z = 1 + iy$, then $w = z^2 = (1 - y^2) + i(2y) = u + iv$. Then the image of S is the set of points $w = u + iv$ with

$$u(x, y) = 1 - y^2 \quad (1)$$

$$v(x, y) = 2y \quad (2)$$

$$\Rightarrow \quad u = 1 - \frac{v^2}{4} \quad (3)$$

since $-\infty < y < \infty \Rightarrow -\infty < v < \infty$. Then, C', i.e. the image of C, is a parabola in the w-plane with vertex at $(1, 0)$, see Fig. 1

Parametric curves: If $x(t)$ and $y(t)$ are real-valued functions of a real variable t, then the set C consisting of all points $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, is called a **parametric curve** of a **complex parametric curve**. The complex valued function of the real variable t, $z(t) = x(t) + iy(t)$, is called a **parametrization** of C.

Line: Let us assume we want to find a parametrization of the line in the complex plane containing the points z_0 and z_1. The vector $z_1 - z_0$ represent a vector originating at z_0 and terminating at z_1. If z is any point on the line containing at z_0 and z_1, then the vector $z - z_0$ is a real multiple of the vector $z_1 - z_0$. Therefore, if z is on the line containing z_0 and z_1, then there is a real number t such that $z - z_0 = t(z_1 - z_0)$. Solving this equation for z gives a

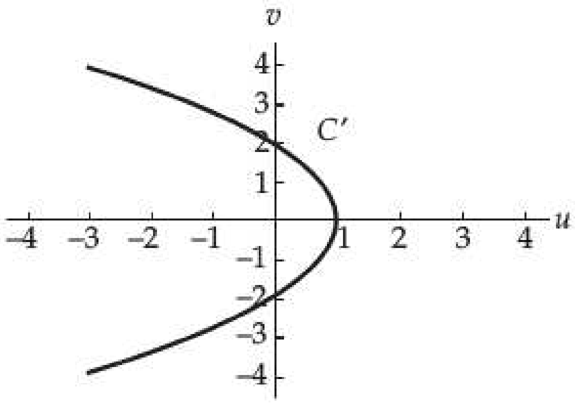

Figure 1: $w = z^2 = (1 - y^2) + i(2y) = u + iv$ (from the book)

parametrization $z(t) = z_0 + t(z_1 - z_0) = z_0(1-t) + z_1 t$. Then, a parametrization of the line containing the points z_0 and z_1 is

$$z(t) = z_0(1-t) + z_1 t \tag{4}$$

with $-\infty < t < \infty$ (for $t \in [0,1]$, $z \in [z_0, z_1]$).

Circle: A parametrization of the circle centered at z_0 with radius r is

$$\begin{aligned} z(t) &= z_0 + r(\cos t + i \sin t) && (5) \\ &= z_0 + re^{it} && (6) \end{aligned}$$

with $0 \le t \le 2\pi$ ($0 \le t \le \pi$ gives a semicircular arc).

Image of a Parametric Curve under a Complex Mapping: If $w = f(z)$ is a complex mapping and if C is a curve parametrized by $z(t)$, $a \le t \le b$, then

$$w(t) = f(z(t)), \tag{7}$$

with $a \le t \le b$ is a parametrization of the image, C' of C under $w = f(z)$.

Example: Image of the semicircle center at zero of $r = 2$ contained in $\mathcal{C}$ under the mapping $w = z^2$.
The points $z \in C$ can be parametrized by $z = 2e^{it}$ with $0 \le t \le \pi$, then, the points $w \in C'$ verify $w(t) = (2e^{it})^2 = 4e^{2it}$ with $0 \le t \le \pi$, i.e. a circle of radius $r' = 4$, since for $s = 2t$, $w(s) = 4e^{is}$ with $0 \le s \le 2\pi$.

Linear Mappings

Every nonconstant complex linear mapping can be described as a composition of three basic types of motions: a translation (T), a rotation (R), and a magnification (M).

Translations

A complex linear function $T(z) = z + b$ with $b \neq 0$, is called a translation. The linear mapping $T(z) = z + b$ can be visualized in a single copy of the complex plane (single copy means that both, z and $w = T(z)$ are graph in the same complex plane) as the process of *translating* the point z along the vector b with $b = x_0 + iy_0$ to the point $T(z)$. Then, the mapping $T(z) = z + b$ is also called a *translation by b*. A translation does not change the shape or size of a figure in the complex plane

Example: Find the image S' of the square S with vertices

$$z_1 = 1 + i \tag{8}$$
$$z_2 = 2 + i \tag{9}$$
$$z_3 = 2 + 2i \tag{10}$$
$$z_4 = 1 + 2i \tag{11}$$

under the linear mapping $T(z) = z + 2 - i$.
Solution: From $T(z) = z + 2 - i = z + b$ with $b = 2 - i$, we get that each point on the square S will translate by the vector $(2, -1)$ in the complex plane. In particular the vertices will be translate to

$$T(z_1) = 3 \tag{12}$$
$$T(z_2) = 4 \tag{13}$$
$$T(z_3) = 4 + i \tag{14}$$
$$T(z_4) = 3 + i \tag{15}$$

Rotation

A complex linear function

$$R(z) = az \tag{16}$$

with $|a| = 1$ is called a **rotation**. Written in polar we get

$$R(z) = e^{i\theta} re^{i\phi} = re^{i(\theta+\phi)} \tag{17}$$

Then, $|R(z)| = r = |z|$ and $arg(R(z)) = \theta + \phi = arg(z) + \theta$. Therefore, the linear mapping $R(z) = az$ can be visualized in a single copy of the complex plane as the process of *rotating* the point z counterclockwise through an angle of θ radians about the origin to the point $R(z)$. If $Arg(a) < 0$, then the linear mapping $R(z) = az$ can be visualized in a single copy of the complex plane as the process of rotating points clockwise through an angle of θ radians about the origin. For this reason the angle $\theta = Arg(a)$ is called an angle of rotation of R.

Example: Find the image of C, where C is the real axis $y = 0$ under the linear mapping

$$R(z) = az = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right) z$$

The modulus and principal argument of a are

$$|a| = \left| \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right| = \frac{2}{4} + \frac{2}{4} = 1 \tag{18}$$

$$Arg(a) = \tan^{-1} \left(\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \right) = \tan^{-1}(1) = \frac{\pi}{4} \tag{19}$$

Magnifications

A complex function

$$M(z) = az$$

with $a > 0$ (with $a \in \mathbb{R}$) is called a **magnification**,

$$\begin{aligned} M(z) = az &= (ax) + i(ay) && (20) \\ &= (ar)e^{i\theta} && (21) \end{aligned}$$

then

$$\begin{aligned} |M(z)| &= |a||z| && (22) \\ arg(M(z)) &= arg(z) && (23) \end{aligned}$$

Thus, the linear mapping $M(z) = az$ can be visualized in a single copy of the complex plane as the process of *magnifying*, for $a > 1$ the modulus of the point z by a factor of a to obtain the point $M(z)$. The real number a is called the **magnification factor** of M. If $0 < a < 1$, then the point $M(z)$ is a times closer to the origin than the point z. This special case of a magnification is called a **contraction**.

A magnification mapping will change the size of a figure but, it will not change its basic shape.

Example: Image of the circle C given by $|z| = 2$ under the linear mapping $M(z) = 3z$.
Solution: Each point in the image C' will have modulus $|M(z)| = |3z| = 6$. The image points can have any argument since the points z in the circle C can have any argument. Therefore, the image C' is the circle $|w| = 6$ that is centered at the origin and has radius 6.

Linear Mappings

A general linear mapping $f(z) = az + b$ is a composition of a rotation, a magnification, and a translation.

Image of a Point under a Linear Mapping: Let us suppose that

$$f(z) = az + b = |a|\left(\frac{a}{|a|}z\right) + b$$

is a complex linear function with $a \neq 0$ and z_0 be a point in the complex plane. If the point $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by

1. the term $a/|a|$ rotates z_0 an angle $Arg(a)$ about the origin,
2. the term $|a|$ is a magnification with magnification factor $|a|$, and
3. the term b translates the result by b.

This description also describes the image of any set of points S. In particular, the image, S', of a set S under $f(z) = az + b$ is the set of points obtained by rotating S through $Arg(a)$, magnifying by $|a|$, and then translating by b.

From $f(z) = |a|(a/|a|)z + b$ we see that every nonconstant complex linear mapping is a composition of *at most* one rotation, one magnification, and one translation. Then, if $a \neq 0$ is a complex number, $R(z)$ is a rotation through $Arg(a)$, $M(z)$ is a magnification by $|a|$, and $T(z)$ is a translation by b, then the composition $f(z) = (T \circ M \circ R)(z) = T(M(R(z)))$ is a complex linear function.

Since the composition of any finite number of linear functions is again a linear function, it follows that the composition of finitely many rotations, magnifications, and translations is a linear mapping.

A linear mapping $f(z) = az + b$ with $a \neq 0$ can distort the size of a figure in the complex plane, but it *preserve the basic shape* of a figure.

The order of the composition is important (in some special cases the changing the order does not change the mapping, see problems 27 and 28 in the book). Example where the result change: consider the mapping $f(z) = 2z + i$, which magnifies by 2, then translates by i; so, $f(0) = i$. If we reverse the order of composition, that is, if we translate by i, then magnify by 2 the effect is 0 maps onto $2i$!!.

A complex linear mapping can always be represented as a composition in more than one way. The complex mapping $f(z) = 2z + i$, for example, can also be expressed as $f(z) = 2(z + i/2)$.

Example: Find the image of the rectangle with vertices

- $-1 + i$,
- $1 + i$,
- $1 + 2i$,
- $-1 + 2i$

under the linear mapping $f(z) = 4iz + 2 + 3i$.

Solution 1: Let S be the rectangle with the above vertices and let S' denote the image of S under f. Because f is a linear mapping, our foregoing discussion implies that S' has the same shape as S. That is, S' is also a rectangle with vertices

- $f(-1 + i) = -2 - i$,
- $f(1 + i) = -2 + 7i$,
- $f(1 + 2i) = -6 + 7i$,
- $f(-1 + 2i) = -6 - i$.

Solution 2: The linear mapping f can also be viewed as a composition of a

- rotation $Arg(4i) = \pi/2$
- magnification $|4i| = 4$
- translation $2 + 3i$

see Fig. 2.

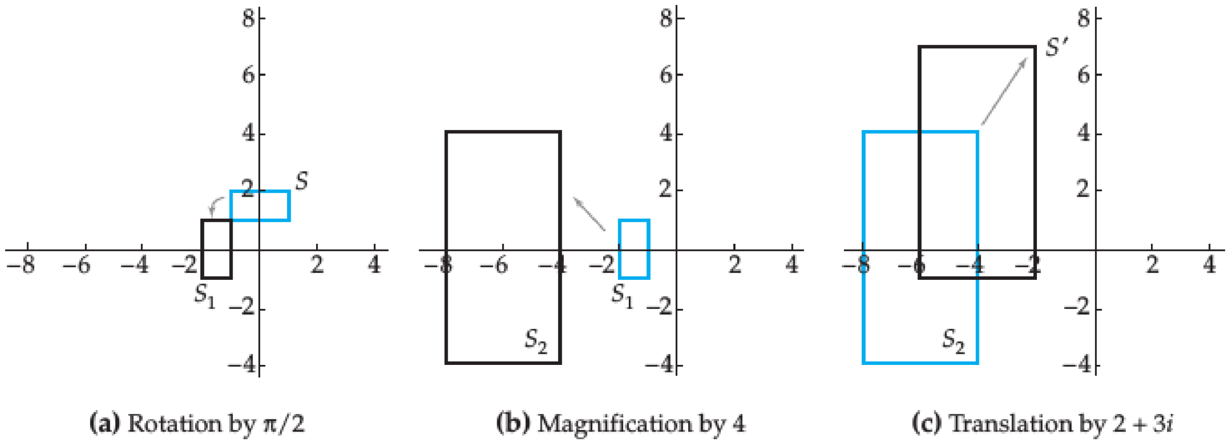

Figure 2: Linear mapping $f(z) = 4iz + 2 + 3i$ (from the book)

Special Power Functions

A **complex polynomial function** is a function of the form $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where n is a positive integer and $a_n, a_{n-1}, \cdots, a_1, a_0$ are complex constants. In this section we study complex polynomials of the form $f(z) = z^n$, $n \geq 2$. The mappings $w = z^n$, with $n \geq 2$, do not preserve the basic shape of every figure in the complex plane. Associated to the function z^n, $n \geq 2$, we also have the *principal nth root function* $z^{1/n}$. The principal nth root functions are inverse functions of the functions z^n defined on a sufficiently restricted domain.

Power Functions A complex power function is a function of the form $f(z) = z^\alpha$ where α is a complex constant. In this section we will restrict our attention to special complex power functions of the form z^n and $z^{1/n}$ where $n \geq 2$ and n is an integer.

The function z^n

The function z^2: The values of $f(z) = z^2$ are found using complex multiplication. For example, at $z = 2 - i$, we get $f(2-i) = (2-i)^2 = (2-i)(2-i) = 3 - 4i$. Understanding the complex mapping $w = z^2$, requires a little more work. We begin by expressing this mapping in exponential notation by replacing the symbol z with $re^{i\theta}$: $w = z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$. Then, $|w| = r^2$ and $arg(w) = 2\theta = 2arg(z)$, see Fig. 3.

Mapping $\mathbb{R}^+$ by z^2: The squaring function z^2 maps a semicircle $|z| = r$, $-\pi/2 \leq arg(z) \leq \pi/2$, onto a circle $|w| = r^2$, $-\pi \leq arg(w) \leq \pi$. Then, since the right half-plane $Re(z) \geq 0$ consists of the collection of semicircles $|z| = r$, $-\pi/2 \leq arg(z) \leq \pi/2$, with $r \in [0, \infty)$, we have that the image of this half-plane consists of the collection of circles $|w| = r^2$ where r takes on any value in $[0, \infty)$. This implies that $w = z^2$ maps the right half-plane $Re(z) \geq 0$ onto the entire complex plane.

Mapping a triangle by z^2: Let us find the image of the triangle S with vertices

- $z_1 = 0$

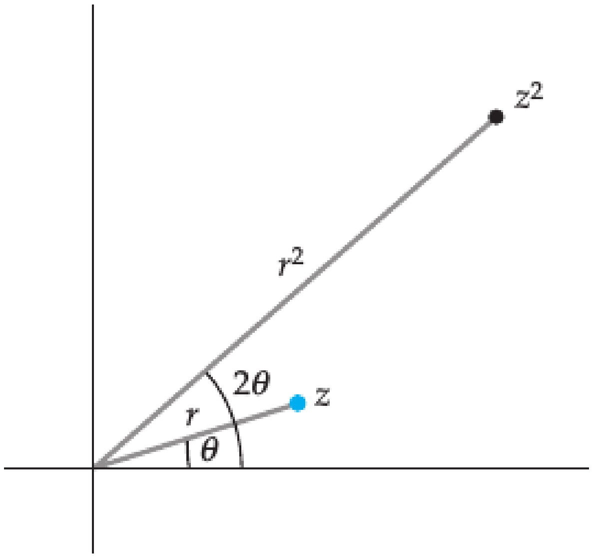

Figure 3: The mapping $w = z^2$ (from the book)

- $z_2 = 1 + i$
- $z_3 = 1 - i$

Solution: Let us denote S' the image of S under $w = z^2$. Each of the three sides of S will be treated separately.

1. The side of $z_1 - z_2$ lies on a ray emanating from the origin and making an angle of $\pi/4$ with the positive x-axis. The image of this segment must lie on a ray making an angle of $2(\pi/4) = \pi/2$ with the positive u-axis. Since the module of the points on the edge containing 0 and $1 + i$ vary from 0 to 2, the moduli of the images of these points vary from $0^2 = 0$ to $(\sqrt{2})^2 = 2$. Thus, the image of $z_1 - z_2$ is a vertical line segment from 0 to $2i$.

2. In a similar manner, the image of the side of S containing the vertices $z_1 = 0$ and $z_3 = 1-i$ is a vertical line segment from 0 to $-2i$.

3. The remaining side of S contains the vertices $z_3 = 1 - i$ and $z_2 = 1 + i$ consists of the set of points $z = 1 + iy$, $-1 \leq y \leq 1$. Because this side is contained in the vertical line $x = 1$, it follows from a previous example that its image is a parabolic segment given by $u = 1 - v^2/4$, $-2 \leq v \leq 2$ or $v = \pm 2\sqrt{1-u}$, $1 \leq u \leq 0$.

The function z^n, $n > 2$: If z and $w = z^n$ are plotted in the same copy of the complex plane, then this mapping can be visualized as the process of magnifying or contracting the modulus r of $z = re^{i\theta}$ to the modulus r^n of w, and by rotating z about the origin to increase an argument θ of z to an argument $n\theta$ of w,

$$w = z^n = r^n e^{in\theta}$$

.

The power function $z^{1/n}$

The n roots of a nonzero complex number

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

are given by:

$$\sqrt[n]{r}e^{i(\theta+2k\pi)/n} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$$

where $k = 0, 1, 2, \cdots, n-1$.

Principal Square Root Function $z^{1/2}$: For $n = 2$, the two roots of a nonzero complex number

$$\sqrt{r}e^{i(\theta+2k\pi)/2} = \sqrt{r}\left(\cos\frac{\theta+2k\pi}{2} + i\sin\frac{\theta+2k\pi}{2}\right)$$

where $k = 0, 1$.

This equation does not define a function because it assigns two complex numbers (one for $k = 0$ and one for $k = 1$) to the complex number z. By setting $\theta = Arg(z)$ and $k = 0$ we can define a function that assigns to z the unique square root called **principal square root function**, given by

$$z^{1/2} = \sqrt{|z|}e^{iArg(z)/2}$$

If we set $\theta = Arg(z)$ and replace z with $re^{i\theta}$ in $z^{1/2} = \sqrt{|z|}e^{iArg(z)/2}$ we get $z^{1/2} = \sqrt{r}e^{i\theta/2}$.

Examples: The following are the principal square root $z^{1/2}$ for

- $z = 4 \Rightarrow z^{1/2} = \sqrt{|z|}e^{iArg(z)/2} = \sqrt{4}e^{i0/2} = 2$
- $z = -2i \Rightarrow z^{1/2} = \sqrt{|z|}e^{iArg(z)/2} = \sqrt{2}e^{i(-\pi/2)/2} = \sqrt{2}e^{-i\pi/4} = 1 - i$
- $z = -1 + i \Rightarrow |z| = \sqrt{2}$, $\tan\theta = 1/(-1) = -1 \Rightarrow \theta = 3\pi/4$, then $z^{1/2} = \sqrt{|z|}e^{iArg(z)/2} = \sqrt{\sqrt{2}}e^{i(3\pi/4)/2} = \sqrt[4]{2}e^{i3\pi/8}$
- $z = i \Rightarrow i^{1/2} = \sqrt{1}e^{i(\pi/2)/2} = e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$.

Inverse Function

One-to-one: A complex function f is one-to-one if each point w in the range of f is the image of a unique point z, called the pre-image of w, in the domain of f. That is, f is one-to-one if whenever $f(z_1) = f(z_2)$, then $z_1 = z_2$. This says that a one-to-one complex function will not map distinct points in the z-plane onto the same point in the w-plane. If f is a one-to-one complex function, then for any point w in the range of f there is a unique pre-image in the z-plane, which we denote by $f^{-1}(w)$. This correspondence between a point w and its pre-image $f^{-1}(w)$ defines the inverse function of a one-to-one complex function.

Inverse Function: If f is a one-to-one complex function with domain A and range B, then the inverse function of f, denoted by f^{-1}, is the function with domain B and range A defined by $f^{-1}(z) = w$ if $f(w) = z$.

Then, if a set S is mapped onto a set S' by a one-to-one function f, then f^{-1} maps S' onto S. In other words, the complex mappings f and f^{-1} 'undo' each other. It also follows that if f has an inverse function, then $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$.

Example: inverse function of z^2 Inverse Functions of z^n, $n \geq 2$ is not well defined since it is not one-to-one. In order to see that this is so, consider the points $z_1 = re^{i\theta}$ and $z_2 = re^{i(\theta+2\pi/n)}$ with $r \neq 0$. Because $n \geq 2$, the points z_1 and z_2 are distinct but $f(z_1) = r^n e^{in\theta}$ and $f(z_2) = r^n e^{i(n\theta+2\pi)} = r^n e^{in\theta} e^{i2\pi} = r^n e^{in\theta} = f(z_1)$.

For $n = 2$ the function $f(z) = z^2$ is a one-to-one function on the set A defined by $-\pi/2 < Arg(z) \leq \pi/2$. It follows that this function has a well-defined inverse function f^{-1}. This inverse function is the principal square root function $z^{1/2}$.

Remember: the principal argument $Arg(z)$ of a complex number z lies $-\pi < Arg(z) \leq \pi$

Let $z = re^{i\theta}$ and $f^{-1} = w = \rho e^{i\phi}$ where $\theta = Arg(z)$ $(-\pi < Arg(z) \leq \pi)$ and $\phi = Arg(w)$. Since the range of $f^{-1} = z^{1/2}$ is the domain of $f = z^2$, then $-\pi/2 < \phi \leq \pi/2$ and $\rho = \sqrt{r}$ (ver figure 4), then

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \tag{24}$$

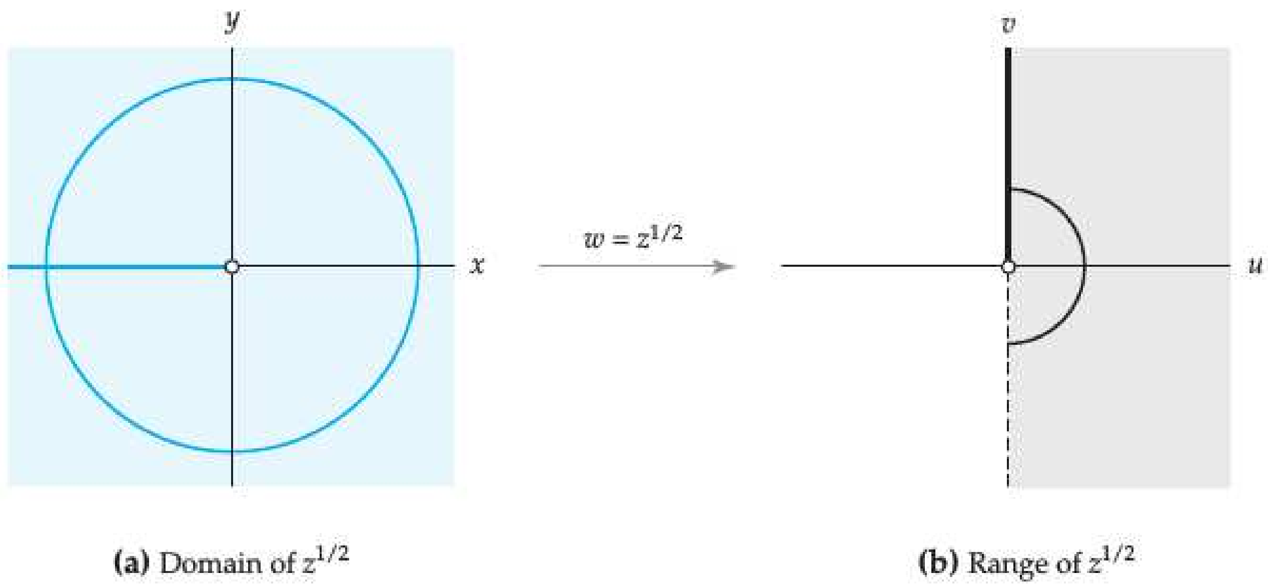

Figure 4: The principal square root function $w = z^{1/2}$ (from the book)

The Mapping $w = z^{1/2}$ As a mapping, the function z^2 squares the modulus of a point and doubles its argument. Because the principal square root function $z^{1/2}$ is an inverse function of z^2, it follows that the mapping $w = z^{1/2}$ takes the square root of the modulus of a point and halves its principal argument. That is, if $w = z^{1/2}$, then we have $|w| = \sqrt{|z|}$ and $Arg(w) = \frac{1}{2}Arg(z)$.

Principal nth Root Function

By modifying the argument given for the function $f(z) = z^2$ is one-to-one on the set defined by $-\pi/2 < arg(z) \leq \pi/2$, it can be show that the complex power function $f(z) = z^n$, $n > 2$, is one-to-one on the set defined by

$$-\frac{\pi}{n} < arg(z) \leq \frac{\pi}{n} \tag{25}$$

The image of the set defined by above under the mapping $w = z^n$ is the entire complex plane $\mathcal{C}$ excluding $w = 0$. Therefore, there is a well-defined inverse function for f. Analogous to the case $n = 2$, this inverse function of z^n is called the **principal nth root function** $z^{1/n}$.

The domain of $z^{1/n}$ is the set of all nonzero complex numbers, and the range of $z^{1/n}$ is the set of complex numbers w satisfying $-\pi/n < arg(z) \le \pi/n$,

$$z^{1/n} = \sqrt[n]{|z|}e^{iArg(z)/n} = \sqrt[n]{|z|}e^{i\theta/n} \tag{26}$$

with $\theta = Arg(z)$.

Multiple-Valued Functions

A nonzero complex number z has n distinct nth roots in the complex plane. This means that the process of "taking the nth root" of a complex number z does not define a complex function because it assigns a set of n complex numbers to the complex number z. These types of operations on complex numbers are examples of **multiple-valued functions**. We will adopt the following functional notation for multiple-valued functions: (i) when representing multiple-valued functions with functional notation, we will use uppercase letters such as $F(z) = z^{1/2}$ or $G(z) = arg(z)$; (2) lower-case letters such as f and g will be reserved to represent functions, for example, $f(z) = z^{1/2}$ refers to the principal square root function.

The visualization of multiple-to-one a complex mapping like z^2 is the **Riemann surface**. See Fig. 5 (it seems that the cut in the Riemann surface is in $\mathbb{R}^+$ while the mapping has the cut at $\mathbb{R}^-$.)

Reciprocal Function

We define a complex rational function to be a function of the form $f(z) = p(z)/q(z)$ where both $p(z)$ and $q(z)$ are complex polynomial functions. In this section, we study the most basic complex rational function, the reciprocal function $1/z$, as a mapping of the complex plane. An important property of the reciprocal mapping is that it maps certain lines onto circles.

The function $1/z$, whose domain is the set of all nonzero complex numbers, is called the **reciprocal function**,

$$w = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \tag{27}$$

therefore, the reciprocal function maps a point in the z-plane with polar coordinates (r, θ) onto a point in the w-plane with polar coordinates $(1/r, -\theta)$, see Fig. 6

Inversion in the Unit Circle The function

$$g(z) = \frac{1}{r}e^{i\theta}$$

whose domain is the set of all nonzero complex numbers, is called inversion in the unit circle.

We will describe this mapping by considering separately

1. the images of points on the unit circle,
2. the points outside the unit circle, and
3. the points inside the unit circle.

(1) Consider first a point z on the unit circle. Since $z = 1e^{i\theta}$, then $g(z) = e^{i\theta} = z$. Therefore, each point on the unit circle is mapped onto itself by g.
(2) If z is a nonzero complex number that does not lie on the unit circle, then we can write z as $z = re^{i\theta}$ with $r \ne 1$. When $r > 1$ (z is outside of the unit circle), we have that $|g(z)| = 1/r < 1$.

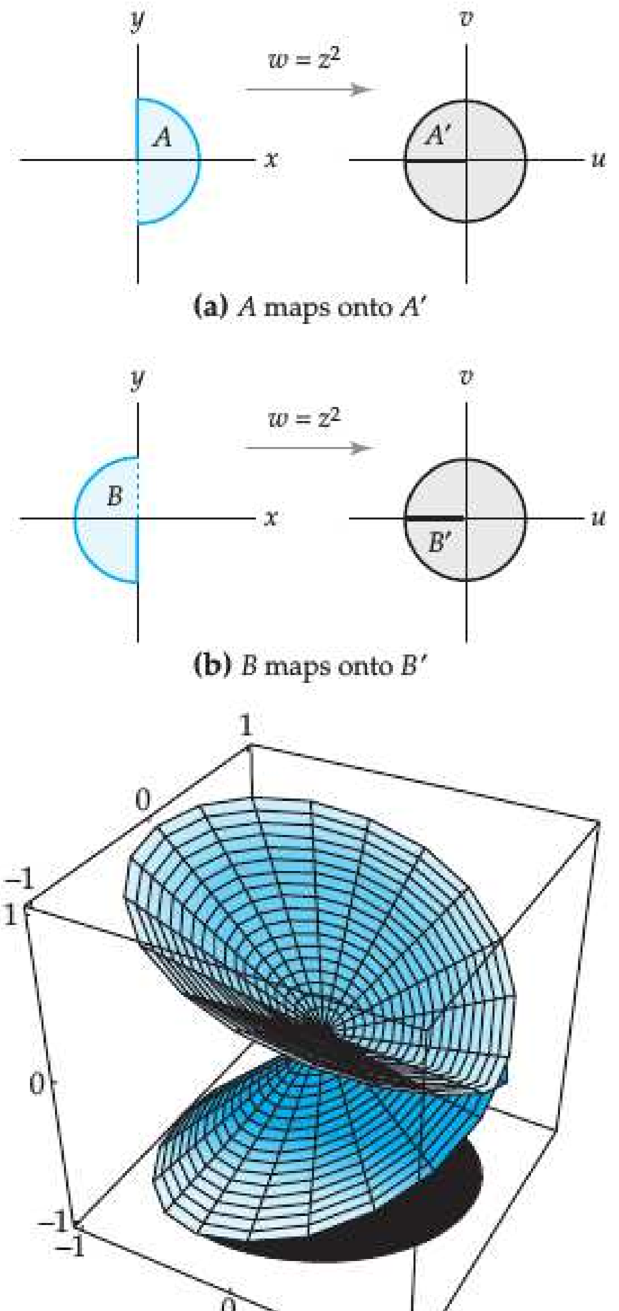

Figure 5: Two mapping for $w = z^2$ and a Riemann surface for the one-to-one valued function $f(z) = z^2$ (from the book)

So, the image under g of a point z outside the unit circle is a point inside the unit circle.
(3) Conversely, if $r < 1$, then $|g(z)| = 1/r > 1$, and we conclude that if z is inside the unit circle, then its image under g is outside the unit circle.
See Fig. 7

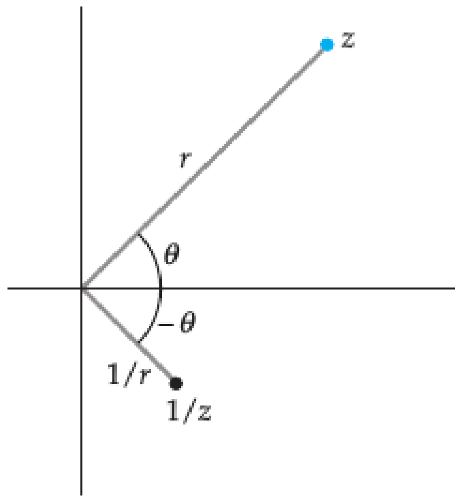

Figure 6: The reciprocal mapping for $w = 1/z$ (from the book).

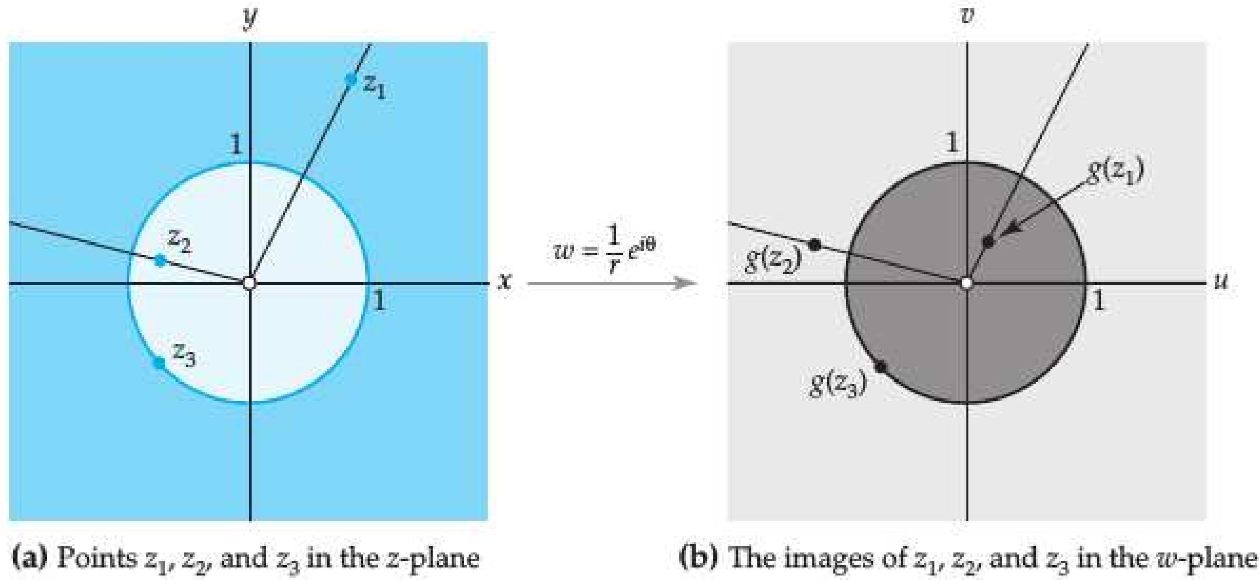

Figure 7: Inversion in the unit circle $g(z) = \frac{1}{r}e^{i\theta}$ (from the book).

Complex Conjugation

The second complex mapping that is helpful for describing the reciprocal mapping is a reflection across the real axis. Under this mapping the image of the point (x, y) is $(x, -y)$, $c(z) = \bar{z}$, called **complex conjugation function**,

$$c(z) = c(x + iy) = x - iy \tag{28}$$
$$c(z) = c(re^{i\theta}) = re^{-i\theta} \tag{29}$$

Reciprocal Mapping

The reciprocal function $f(z) = 1/z$ can be written as the composition of inversion in the unit circle and complex conjugation. Using the exponential forms $c(z) = re^{-i\theta}$ and $g(z) = e^{i\theta}/r$ of these functions we find that the composition $c \circ g$ is given by:

$$c(g(z)) = c\left(\frac{1}{r}e^{i\theta}\right) = \frac{1}{r}e^{-i\theta}$$

Then $c(g(z)) = f(z) = 1/z$. This implies that, as a mapping, the reciprocal function first inverts in the unit circle, then reflects across the real axis.

Image of a Point under the Reciprocal Mapping Let z_0 be a nonzero point in the complex plane. If the point $w_0 = f(z_0) = 1/z_0$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by: (i) inverting z_0 in the unit circle, then, (ii) reflecting the result across the real axis.

Image of a semicircle: Let us find the image of the semicircle $|z| = 2$, $0 \leq arg(z) \leq \pi$ under $w = 1/z$.
Solution: the mapping is the semicircle $|w| = 1/2$, $-\pi \leq arg(w) \leq 0$. See Fig. 8

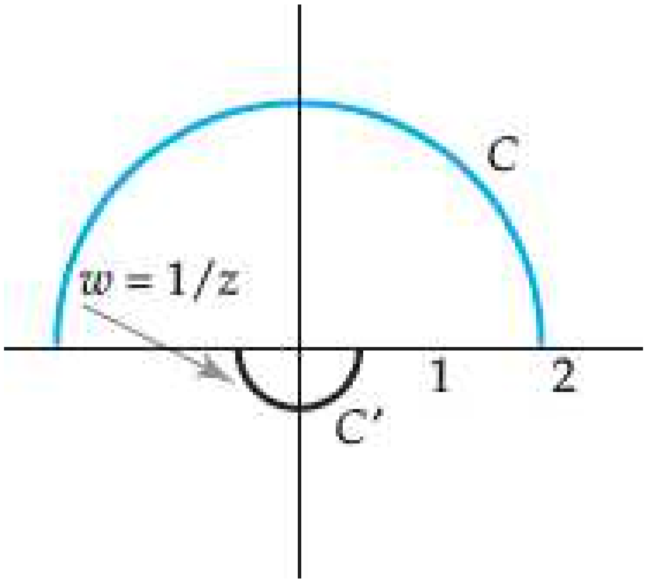

Figure 8: Reciprocal mapping of $|z| = 2$, $0 \leq arg(z) \leq \pi$ (from the book).

Image of a line Let us find the image of the vertical line $x = 1$ under the reciprocal mapping $w = 1/z$.
Solution: The vertical line $x = 1$ consists of the set of points $z = 1 + iy$, $-\infty < y < \infty$, then

$$w = \frac{1}{1+iy} = \frac{1}{1+y^2} - i\frac{y}{1+y^2} = u + iv \Rightarrow u^2 - u + v^2 = 0 \tag{30}$$

then,

$$\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4} \tag{31}$$

with $u \neq 0$ since $v = -yu$. Figure 9 shows the mapping.

Comments:

1) The above equation defines a circle centered at $(1/2, 0)$ with radius $1/2$. However, because

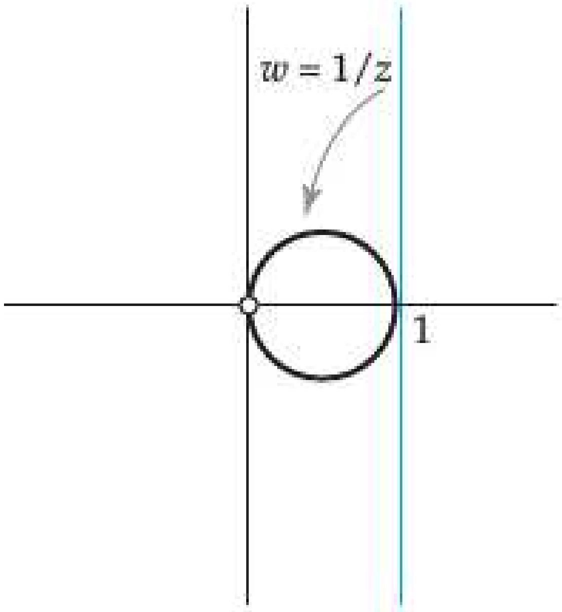

Figure 9: Reciprocal mapping of $x = 1$ (from the book).

$u \neq 0$, the point $(0, 0)$ is not in the image.
2) Using the complex variable $w = u + iv$, we can describe this image by $|w - 1/2| = 1/2$, $w \neq 0$.
3) The point $(0, 0)$ is not included because there is no point on the line $x = 1$ that actually maps onto 0. In order to obtain the entire circle as the image of the line we must consider the reciprocal function defined on the extended complex-number system.
Remember: the extended complex-number system consists of all the points in the complex plane adjoined with the ideal point ∞
4) Points in the extended complex plane that are near the ideal point ∞ correspond to points with extremely large modulus in the complex plane.

Reciprocal Function on the Extended Complex Plane: The reciprocal function on the extended complex plane is the function defined by:

$$f(z) = \begin{cases} 1/z & z \neq 0 \ \ or \ \ z \neq \infty \\ \infty & z = 0 \\ 0 & z = \infty \end{cases} \tag{32}$$

Mapping Lines to Circles with $w = 1/z$ The reciprocal function on the extended complex plane maps:

(i) the vertical line $x = k$ with $k \neq 0$ onto the circle

$$\left| w - \frac{1}{2k} \right| = \left| \frac{1}{2k} \right| \tag{33}$$

(ii) the horizontal line $x = k$ with $k \neq 0$ onto the circle

$$\left| w + i\frac{1}{2k} \right| = \left| \frac{1}{2k} \right| \tag{34}$$

See Fig. 10.

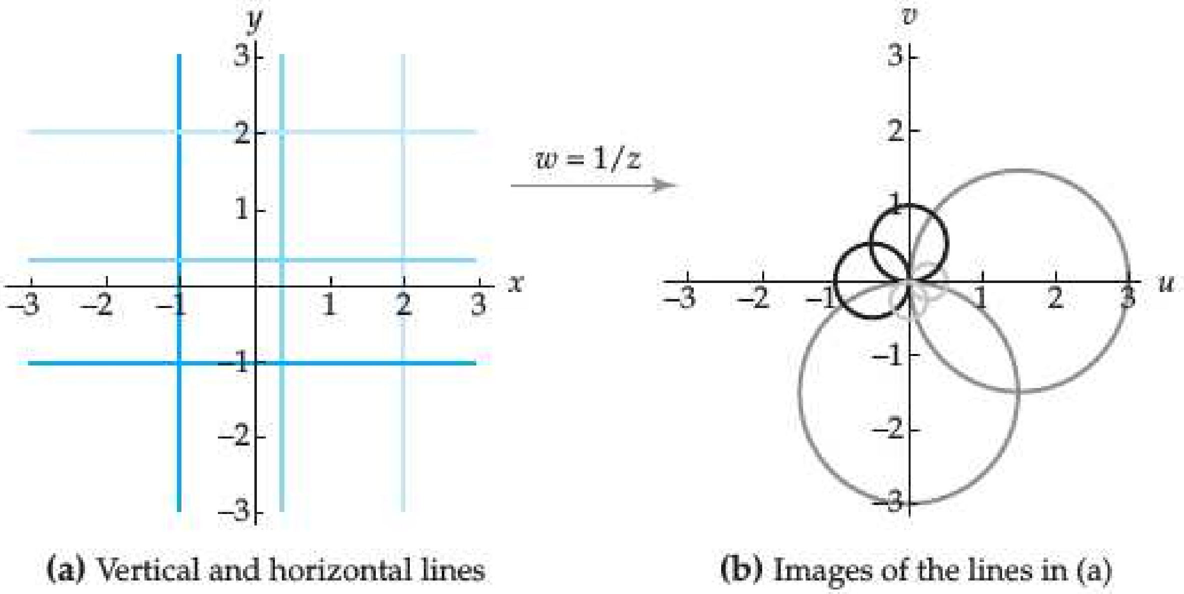

Figure 10: Reciprocal mapping of vertical and horizontal lines (from the book).

Remarks: It is easy to verify that the reciprocal function $f(z) = 1/z$ is one-to-one. Therefore, f has a well-defined inverse function f^{-1}. We find a formula for the inverse function $f^{-1}(z)$ by solving the equation $z = f(w)$ for w. Clearly, this gives $f^{-1}(z) = 1/z$. This observation extends our understanding of the complex mapping $w = 1/z$. For example, we have seen that the image of the line $x = 1$ under the reciprocal mapping is the circle $|w - 1/2| = 1/2$. Since $f^{-1}(z) = 1/z = f(z)$, it then follows that the image of the circle $|z - 1/2| = 1/2$ under the reciprocal mapping is the line $u = 1$. In a similar manner, we see that the circles $|w-1/(2k)| = |1/(2k)|$ and $|w+i/(2k)| = |1/(2k)|$ are mapped onto the lines $x = k$ and $y = k$, respectively.

Limits and Continuity

Remember: recall that $\lim_{x\to x_0} f(x) = L$ *intuitively means that values* $f(x)$ *of the function* f *can be made arbitrarily close to the real number* L *if values of* x *are chosen sufficiently close to, but not equal to, the real number* x_0*.*

The concept of a complex limit is similar to that of a real limit in the sense that $\lim_{z\to z_0} f(z) = L$ will mean that the values $f(z)$ of the complex function f can be made arbitrarily close the complex number L if values of z are chosen sufficiently close to, but not equal to, the complex number z_0.

In this section we will define the limit of a complex function, examine some of its properties, and introduce the concept of continuity for functions of a complex variable.

Limits

Limit of a Complex Function Suppose that a complex function f is defined in a deleted neighborhood of z_0 and suppose that L is a complex number. The limit of f as z tends to z_0 exists and is equal to L, written as $\lim_{z\to z_0} f(z) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. See Fig. 11

Remember: (i) the set of points w in the complex plane satisfying $|w - L| < \varepsilon$ is called a ***neighborhood*** *of L and (ii) the set of points satisfying the inequalities $0 < |z - z_0| < \delta$ is called a* ***deleted neighborhood*** *of z_0*

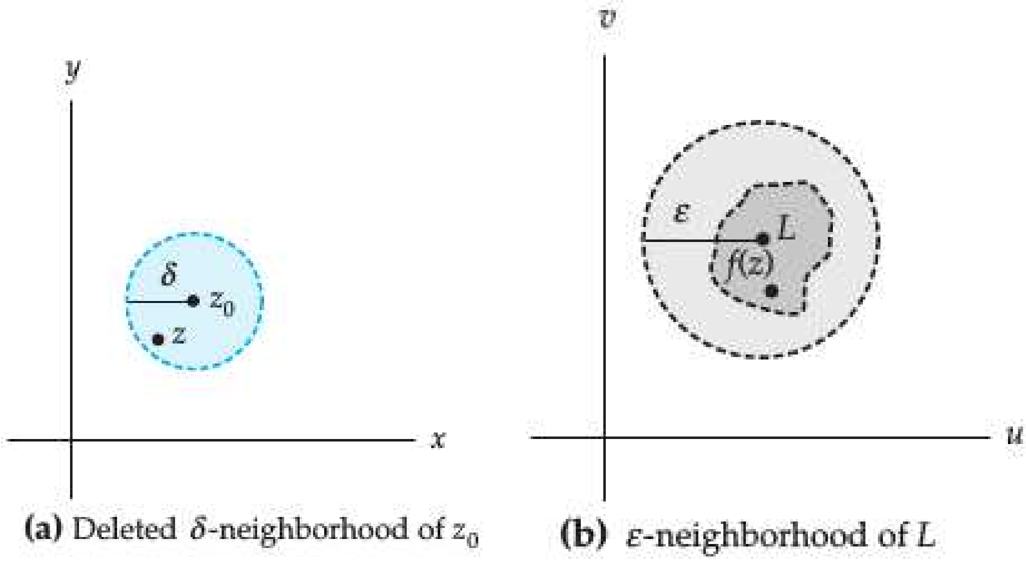

Figure 11: The geometric meaning of a complex limit (from the book).

For limits of complex functions, z is allowed to approach z_0 from any direction in the complex plane, that is, along any curve or path through z_0. In order that $\lim_{z\to z_0} f(z)$ exists and equals L, we require that $f(z)$ approach the same complex number L along every possible curve through z_0.

Criterion for the Nonexistence of a Limit: If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0, then $\lim_{z\to z_0} f(z)$ does not exist.

Example: The limit $\lim_{z\to 0} z/\bar{z}$ does not exit since,

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{z\to 0} \frac{x+i0}{\overline{x+i0}} = \lim_{x\to 0} \frac{x}{x} = 1 \tag{35}$$

$$\lim_{z\to 0} \frac{z}{\bar{z}} = \lim_{z\to 0} \frac{0+iy}{\overline{0+iy}} = \lim_{y\to 0} \frac{iy}{-iy} = -1 \tag{36}$$

Remark 1: In general, computing values of $lim f(z)$ as z approaches z_0 from different directions can prove that a limit does not exist, but this technique cannot be used to prove that a limit does exist. In order to prove that a limit does exist we must use definition of limit directly. This requires demonstrating that for every positive real number ε there is an appropriate choice of δ that meets the requirements of definition of limit. Such proofs are commonly called "epsilon-delta proofs."

Remark 2: The definition of complex limits not provide a convenient method for computing limits. We will present a practical method for computing complex limits shortly in a theorem. In addition to being a useful computational tool, this theorem also establishes an important connection between the complex limit of $f(z) = u(x, y) + iv(x, y)$ and the real limits of the real-valued functions of two real variables $u(x, y)$ and $v(x, y)$.

Limit of the Real Function $F(x,y)$: The limit of F as (x,y) tends to (x_0,y_0) exists and is equal to the real number L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|F(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Theorem (2.1): Real and Imaginary Parts of a Limit Suppose that $f(z) = u(x,y) + iv(x,y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then lim $\lim_{z\to z_0} f(z) = L$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \tag{37}$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 \tag{38}$$

Example Let us calculate the following limit: $\lim_{z\to 1+i}(z^2+1)$.
Solution: let us start separating the real and imaginary parts of (z^2+1)

$$\Re(z^2+1) = x^2 - y^2 \tag{39}$$
$$\Im(z^2+1) = 2xy + 1 \tag{40}$$

then

$$\lim_{(x,y)\to(1,1)} (x^2 - y^2) = 0 \tag{41}$$
$$\lim_{(x,y)\to(1,1)} (2xy + 1) = 3 \tag{42}$$

finally

$$\lim_{z\to 1+i} (z^2+1) = i3 \tag{43}$$

Theorem (2.2): Properties of complex limits Suppose that f and g are complex functions. If $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then

1. $\lim_{z\to z_0} cf(z) = cL$, with c a complex constant
2. $\lim_{z\to z_0}(f(z) \pm g(z)) = L \pm M$
3. $\lim_{z\to z_0} f(z)g(z) = LM$
4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L}{M}$, provided $M \neq 0$.

Exercises for the student in class: Calculates the following limits:

1. $lim_{z\to i} \frac{(3+i)z^4 - z^2 + 2z}{z+1} = \frac{7}{2} - i\frac{1}{2}$
2. $lim_{z\to 1+i\sqrt{3}} \frac{z^2 - 2z + 4}{z - 1 - i\sqrt{3}} = i2\sqrt{3}$

Continuity

Continuity of a Complex Function

A complex function f is continuous at a point z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0) \tag{44}$$

Criteria for Continuity at a Point: A complex function f is continuous at a point z_0 if each of the following three conditions hold:

1. $\lim_{z\to z_0} f(z)$ exits,
2. f is defined at z_0, and
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

Discontinuous at z_0 If a complex function f is not continuous at a point z_0 then we say that f is discontinuous at z_0. For example, the function $f(z) = 1/(1+z^2)$ is discontinuous at $z = i$ and $z = -i$.

Example: The principal square root function $f(z) = z^{1/2}$ is discontinuous at the point $z_0 = -1$ since the limit from upper unit circle gives i while the limit from a lower unit circle gives $-i$.

Continuity in a set: A complex function f is continuous on a set S if f is continuous at z_0 for each z_0 in S. For example the function $f(z) = z^2 - iz + 2$ is continuous at any point z_0 in the complex plane, while the function $g(z) = 1/(z^2+1)$ is continuous on the set consisting of all complex z such that $z \neq \pm i$.

Theorem (2.3): Real and Imaginary Parts of a Continuous Function Suppose that $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$. Then the complex function f is continuous at the point z_0 if and only if both real functions u and v are continuous at the point (x_0, y_0).

Theorem (2.4): Properties of Continuous Functions: If f and g are continuous at the point z_0 , then the following functions are continuous at the point z_0:
(i) cf, c a complex constant,
(ii) $f \pm g$,
(iii) $f \cdot g$, and
(iv) f/g provided $g(z_0) \neq 0$.
(ii) and (iii) extend to any finite sum or finite product of continuous functions, respectively. We can use these facts to show that polynomials are continuous functions.

Theorem (2.5): Continuity of Polynomial Functions Polynomial functions are continuous on the entire complex plane $\mathbb{C}$.

Since a rational function $f(z) = p(z)/q(z)$ is quotient of the polynomial functions p and q, it follows from the above two Theorems that f is continuous at every point z_0 for which $q(z_0) \neq 0$. In other words, i.e. *rational functions are continuous on their domains.*

Bounded functions

Continuous complex functions have many important properties that are analogous to properties of continuous real functions. For instance, recall that if a real function f is continuous on a closed interval I on the real line, then f is bounded on I. This means that there is a real number $M > 0$ such that $|f(x)| \leq M$ for all x in I. An analogous result for real functions $F(x, y)$ states that if $F(x, y)$ is continuous on a closed and bounded region R of the Cartesian plane, then there is a real number $M > 0$ such that $|F(x, y)| \leq M$ for all (x, y) in R, and we say F is bounded on R.

Now suppose that the function $f(z) = u(x, y) + iv(x, y)$ is defined on a closed and bounded region R in the complex plane. We say that the complex f is bounded on R if there exists a real constant $M > 0$ such that $|f(z)| < M$ for all z in R. If f is continuous on R, then Theorem 2.3 tells us that u and v are continuous real functions on R. It follows that the real function $F(x, y) = \sqrt{[u(x, y)]^2 + [v(x, y)]^2}$ is also continuous on R since the square root function is continuous. Because F is continuous on the closed and bounded region R, we conclude that F is bounded on R. That is, there is a real constant $M > 0$ such that $|F(x, y)| \leq M$ for all (x, y) in R. However, since $|f(z)| = F(x, y)$, we have that $|f(z)| \leq M$ for all z in R. In other words, the complex function f is bounded on R. This establishes the following important property of continuous complex functions.

A Bounding Property If a complex function f is continuous on a closed and bounded region R, then f is bounded on R. That is, there is a real constant $M > 0$ such that $|f(z)| \leq M$ for all z in R.

While this result assures us that a bound M exists for f on R, it offers no practical approach to find it. One approach to find a bound is to use the triangle inequality. Another approach to determine a bound is to use complex mappings.

Branches: We are usually interested in computing just one of the values of a multiple-valued function. For example, the function $F(z) = z^{1/n}$ assigns to the input z the set of n roots of z. If we make a choice such that to keep only one function, for example we keep one of the roots of $F(z)$ together with the concept of continuity in mind, then we obtain a function that is called a **branch** of a multiple-valued function. In more rigorous terms, a branch of a multiple-valued function F is a function f_1 that is continuous on some domain and that assigns exactly one of the multiple-values of F to each point z in that domain. The notation for the branches will be lowercase letters with a numerical subscript such as f_1, f_2, and so on.

About the domain of a branch: The requirement that a branch be continuous means that the domain of a branch is different from the domain of the multiple-valued function. For example, the multiple-valued function $F(z) = z^{1/2}$ is defined for all nonzero complex numbers z. Even though the principal square root function $f(z) = z^{1/2}$ does assign exactly one value of F to each input z, f is not a branch of F because f is not continuous at $z_0 = -1$. In order to get a branch of $F(z) = z^{1/2}$ we have to restrict the domain, i.e. $f_1(z) = \sqrt{r}e^{i\theta/2}$, with $-\pi < \theta < \pi$; it is called the **principal branch** of $F(z) = z^{1/2}$.

Branch Cuts and Branch Points Although the multiple-valued function $F(z) = z^{1/2}$ is defined for all nonzero complex numbers $\mathbb{C}$, the principal branch f_1 is defined only on the domain $|z| > 0$, $-\pi < arg(z) < \pi$. In general, a **branch cut** for a branch f_1 of a multiple-valued function F is a portion of a curve that is excluded from the domain of F so that f_1 is continuous on the remaining points. Therefore, the nonpositive real axis is a branch cut for the principal branch f_1 (defined above).

A different branch of F with the same branch cut is given by $f_2(z) = \sqrt{r}e^{i\theta/2}$, $\pi < \theta < 3\pi$. These branches are distinct because for, say, $z = i$ we have $f_1(i) = 0.5\sqrt{2} + i0.5\sqrt{2}$, but $f_2(i) = -0.5\sqrt{2} - i0.5\sqrt{2}$. Notice that if we set $\phi = \theta - 2\pi$, then the branch $f_2 = \sqrt{r}e^{i(\phi+2\pi)/2} = -\sqrt{r}e^{i\phi/2}$, $-\pi < \phi < \pi$, then $f_2 = -f_1$.

Other branches of $F(z) = z^{1/2}$ can be defined in a manner similar to the above ones by using any ray emanating from the origin as a branch cut. For example, $f_3(z) = \sqrt{r}e^{i\theta/2}$, $-3\pi/4 < \theta < 5\pi/4$, defines a branch of $F(z) = z^{1/2}$. The branch cut for f_3 is the ray $arg(z) = -3\pi/4$ together with the point $z = 0$.

It is not a coincidence that the point $z = 0$ is on the branch cut for f_1, f_2, and f_3. The point $z = 0$ must be on the branch cut of every branch of the multiple-valued function $F(z) = z^{1/2}$. In general, a point with the property that it is on the branch cut of every branch is called a **branch point** of F.

Alternatively, a **branch point** is a point z_0 with the following property: If we traverse any circle centered at z_0 with sufficiently small radius starting at a point z_1, then the values of any branch do not return to the value at z_1. For example, consider any branch of the multiple-valued function $G(z) = arg(z)$. At the point, say, $z_0 = 1$, if we traverse the small circle $|z - 1| = \varepsilon$ counterclockwise from the point $z_1 = 1 - i\varepsilon$, then the values of the branch increase until we reach the point $1 + i\varepsilon$; then the values of the branch decrease back down to the value of the branch at $z_1 = 1 - i\varepsilon$. See Figure 12. This means that the point $z_0 = 1$ is not a branch point.

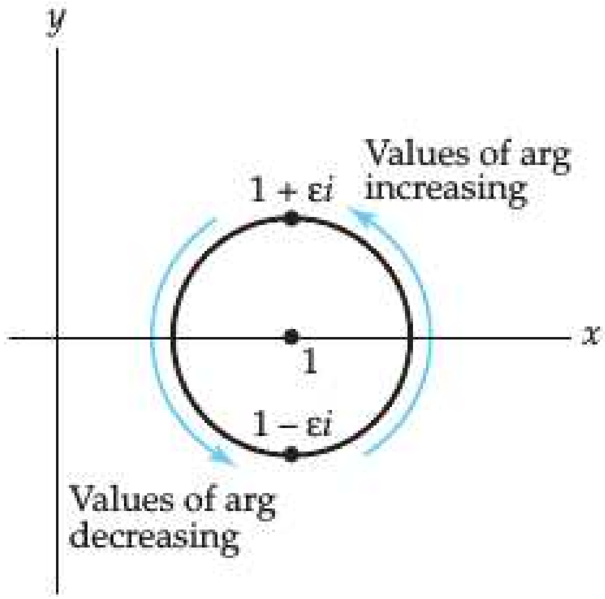

Figure 12: $z = 1$ is not a branch point (from the book)

On the other hand, suppose we repeat this process for the point $z_0 = 0$. For the small circle $|z| = \varepsilon$, the values of the branch increase along the entire circle. See Figures 13. By the time we have returned to our starting point, the value of the branch is no longer the same; it has increased by 2π. Therefore, $z_0 = 0$ is a branch point of $G(z) = arg(z)$.

Limits to infinite and zero: The limit of f as z tends to ∞ exists and is equal to L if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $|z| > 1/\delta$.

Using this definition it is not hard to show that:

$$lim_{z \to \infty} f(z) = L \tag{45}$$

if and only if

$$lim_{z \to 0} f(\frac{1}{z}) = L \tag{46}$$

Infinite limit: The infinite limit

$$lim_{z \to z_0} f(z) = \infty \tag{47}$$

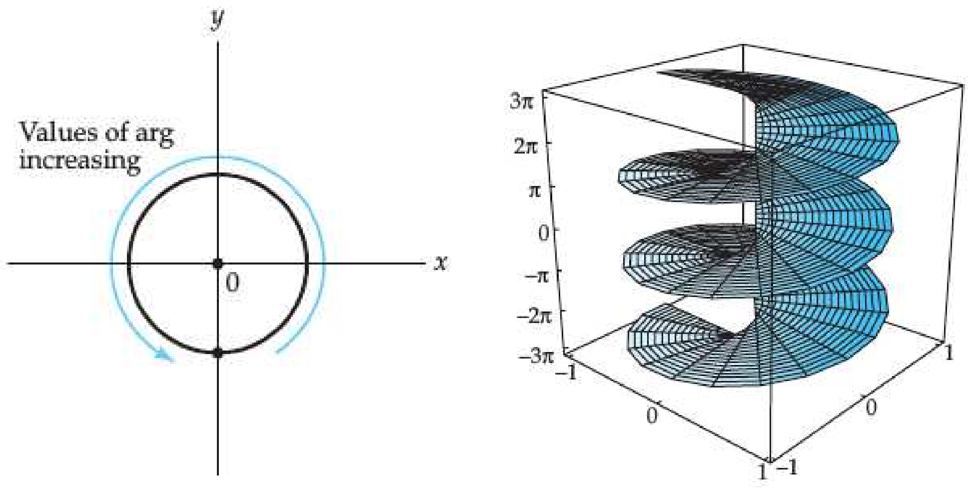

Figure 13: $z = 0$ is a branch point (from the book)

is defined by: The limit of f as z tends to z_0 is ∞ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(z)| > 1/\varepsilon$ whenever $0 < |z - z_0| < \delta$.

From this definition we obtain the following result:

$$lim_{z \to z_0} f(z) = \infty \tag{48}$$

if and only if

$$lim_{z \to z_0} \frac{1}{f(z)} = 0 \tag{49}$$

Continuous complex functions: If a complex function f is continuous on a set S, then the image of every continuous parametric curve in S must be a continuous curve.