Números complejos y plano complejo

Credit: This notes are 100% from chapter 1 of the book entitled *A First Course in Complex Analysis with Applications* by Dennis G. Zill and Patrick D. Shanahan. Jones and Bartlett Publishers. 2003.

In this chapter the complex numbers and some of their algebraic and geometric properties are introduced.

Introduction

Imaginary unit: We say that i is the imaginary unit and define it by the property $i^2 = -1$

Complex number: A complex number is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

Real and imaginary components: a is the real part of the complex number z, while b is its imaginary part, with $Re(z) = a$ and $Im(z) = b$.

Equality: Two complex numbers are equal if their corresponding real and imaginary parts are equal.

Set of complex numbers: The totality of complex numbers is denoted by the symbol $\mathbb{C}$. Since $a \in \mathbb{R}$ can be written as $z = a + i0$ we see that $\mathbb{R}$ is a subset of $\mathbb{C}$.

Arithmetic Operations:

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \tag{1}$$

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2) \tag{2}$$

$$z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2) \tag{3}$$

$$\frac{z_1}{z_2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2} \tag{4}$$

Arithmetic Properties: Commutative laws:

$$z_1 + z_2 = z_2 + z_1 \tag{5}$$

$$z_1z_2 = z_2z_1 \tag{6}$$

Associative laws:

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \tag{7}$$

$$z_1(z_2z_3) = (z_1z_2)z_3 \tag{8}$$

Distributive law:

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \tag{9}$$

Zero and unit: The zero in the complex number system is the number $0 + i0$ and the unit is $1 + i0$ denoted as 0 and 1, respectively.

Additive identity: The zero is the additive identity in the complex number system, i.e. $z + 0 = z$.

Multiplicative identity: The unity is the multiplicative identity, i.e. $z \cdot 1 = z$.

Complex conjugate: The complex conjugate of $z = a + ib$ is defined as $\bar{z} = a - ib$.

Additive inverse: The additive inverse of $z = a + ib$ is $-z = -a - ib$, with $z + (-z) = 0$.

Multiplicative inverse or reciprocal: Every nonzero complex number z has a multiplicative inverse z^{-1} or $1/z$ such that $zz^{-1} = 1$.

Complex Plane

A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y) in a coordinate plane, called complex plane or z-plane, with x the real axis and y the imaginary axis.

Definitions:

Vector: a complex number $z = x + iy$ can be viewed as a two-dimensional position vector.

Modulus or absolute value: the modulus or absolute value of a complex number $z = x+iy$, is the real number $|z| = \sqrt{x^2 + y^2}$, i.e. the distance from the origin to the point (x, y) and $|z|^2 = z\bar{z} = x^2 + y^2$.

Distance: The distance between two points z_1 and z_2 is $|z| = |z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$. From this inequality can be derived these others:

- $|z_1 + z_2| \geq |z_1| - |z_2|$
- $|z_1 + z_2| \geq |(|z_1| - |z_2|)|$
- $|z_1 - z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \geq |(|z_1| - |z_2|)|$
- $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$

Polar Form of Complex Numbers

The complex number $z = x + iy$ has its polar form, or polar representation given by $z = r(\cos\theta + i\sin\theta)$ with $r = |z|$ and θ a positive angle measure in radians from the positive real axis in counterclockwise direction. The parameter θ is called argument of z and denoted by $\theta = arg(z)$, with $\cos\theta = x/r$ and $\sin\theta = y/r$. An argument of a complex number z is not unique since $\cos\theta$ and $\sin\theta$ are 2π-periodic. In practice it is used $\tan\theta = y/x$ to find θ, then if θ_0 is an argument of z, then the angles $\theta_0 \pm 2\pi$, $\theta_0 \pm 4\pi$, $\cdots$ are also arguments of z.

Principal Argument: The argument θ of a complex number that lies in the interval $-\pi < \theta \leq \pi$ is called the principal value of arg(z) or the principal argument of z, and noted as Arg(z), i.e. $-\pi < Arg(z) \leq \pi$. In general, arg(z) and Arg(z) are related by $arg(z) = Arg(z) + 2n\pi$, $n = 0, \pm 1, \cdots$.

Arithmetic Operations:

$$z_1 + z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \quad (10)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \quad (11)$$

$$arg(z_1 z_2) = arg(z_1) + arg(z_2) \quad (12)$$

$$arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) \quad (13)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta) \text{ for any integer } n \quad (14)$$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \text{ for any integer } n \quad (15)$$

Branch cut: If we take arg(z) from the interval $(-\pi, \pi)$, the relationship between a complex number z and its argument is single-valued. For this interval the negative real axis is analogous to a barrier that we agree not to cross; the technical name for this barrier is a branch cut. If we use $(0, 2\pi)$, the branch cut is the positive real axis.

Powers and Roots

There are exactly n solutions of the equation $w^n = z = r(\cos\theta + i\sin\theta)$,

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \quad (16)$$

All of them lie equally spaced $2\pi/n$ on a circle of radius $\sqrt[n]{r}$ centered at the origin in the complex plane, with $k = 0, \cdots, n-1$.

Principal nth Root: $z^{1/n}$ is n-valued, that is, it represents the set of n nth roots w_k of z. The unique root of a complex number z obtained by using the principal value of arg(z) with $k = 0$, is referred as the principal nth root of w.

Rational power: When m and n are positive integers with no common factors, then we can define the rational power of z as $z^{m/n}$, with $(z^{1/n})^m = (z^m)^{1/n} = z^{m/n}$.

Set of Points in the Complex Plane

Circle: The points $z = x + iy$ that satisfy the equation $|z - z_0| = \rho$, $\rho > 0$, lie on a circle of radius ρ centered at the point z_0.

Disk: The points z that satisfy the inequality $|z - z_0| \leq \rho$ can be either *on* the circle $|z - z_0| = \rho$ or *within* the circle. We say that the set of points defined by $|z - z_0| \leq \rho$ is a disk of radius ρ centered at z_0.

Neighborhood: The points z that satisfy the strict inequality $|z - z_0| < \rho$ is called neighborhood of z_0.

Deleted Neighborhood: The neighborhood of z_0 that excludes z_0, i.e. $0 < |z - z_0| < \rho$ is called deleted neighborhood of z_0.

Interior point and Open set: A point z_0 is said to be interior point of a set S of the complex plane if there exist some neighborhood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an open set.

Boundary point: If every neighborhood of a point z_0 of a set S contains at leas one point of S and at least one point not in S, then z_0 is said to be a boundary point of S.

Boundary of a set: The collection of boundary points of a set S is called the boundary of S.

Exterior point: A point z that is neither an interior point nor a boundary point of a set S is said to be an exterior point of S, i.e z_0 is an exterior point of a set S if there exists some neighborhood of z_0 that contains no points of S.

Open circular annulus: The set of points satisfying the simultaneous inequality $\rho_1 < |z - z_0| < \rho_2$ is an open circular ring centered at z_0, called open circular annulus.

Domain: If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segments joined end to end that lies entirely in the set, then the set S is said to be **connected**. An open connected set is called a domain.

Region: a region is a set of points in the complex plane with all, some, none of its boundary points. Since an open set does not contains any boundary points, it is automatically a region.

Closed region: a region that contains all its boundary points is said to be closed. The disk defined by $|z - z_0| \leq \rho$ is an example of a closed region, it is called **closed disk**.

Open region: a neighborhood of a point z_0 defined by $|z - z_0| < \rho$ is an open set or an open region, it is called **open disk**.

Punctured disk: a punctured disk is the region defined by $0 < |z - z_0| \leq \rho$ or $0 < |z - z_0| < \rho$

Bounded Set: a set S in the complex plane is bounded if there exists a real number $R > 0$ such that $|z| < R$ for every z in S.

Unbounded Set: a set is unbounded if it is not bounded.

Extended real-number system: is the set consisting of the real numbers $\mathbb{R}$ adjoined with ∞.

Extended complex-number system: We can associate a complex number with a point on a unit sphere called **Riemmann sphere**. By drawing a line from the number $z = a+ib$, written as $(a, b, 0)$, in the complex plane to the north pole $(0, 0, 1)$ of the sphere $x^2 + y^2 + u^2 = 1$, we determined a unique point (x_0, y_0, u_0) on a unit sphere. In this manner each complex number is identified with a single point on the sphere. Because the point $(0, 0, 1)$ corresponds to no number z in the plane, we correspond it with ∞. The system consisting of $\mathbb{C}$ adjoined with the "ideal point" ∞ is called the extended complex-number system. For a finite number z, we have $z + \infty = \infty + z = \infty$, and for $z \neq 0$, $z \cdot \infty = \infty \cdot z = \infty$. Moreover, for $z \neq 0$ we write $z/0 = \infty$ and for $z \neq \infty$, $z/\infty = 0$. Expressions such as $\infty - \infty$, ∞/∞, ∞^0, and 1^∞ cannot be given a meaningful definition and are called indetermined.