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SOLVING NET CONSTRAINED HYDROTHERMAL NASH-COURNOT EQUILIBRIUM PROBLEMS VIA THE PROXIMAL DECOMPOSITION METHOD

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Abstract: We consider the coupled-in-time Nash-Cournot equilibrium model representing behavior of electricity generating companies acting in an oligopolistic market. Some key features of the model are short-term planning horizon, the possibility for the hydraulic units to pump water back in order to reuse it, and transmission net constraints. Mathematically, it is shown that the problem can be cast as a monotone variational inclusion with some special structure suitable for decomposition. Then the variable metric proximal decomposition method is applied. Some numerical experiments are presented, including a medium-size real-life system. Among other things, it is quantified when the policy of pumping water back can deliver better profits in water-stressed scenarios.

Key words: *electric power market, Nash-Cournot equilibria, variational inequality, inexact proximal point method, decomposition*

Mathematics Subject Classification: *????????????????*

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1 Introduction

In recent decades, the electric power industry has experienced deregulation processes in most of the countries, placing the electricity supply in a market driven structure. Electricity production markets changed from vertically integrated, in which the goal is cost minimization, to a competitive horizontal structure where several actors try to maximize their benefits. This setting is often modeled as an oligopoly, and the key issue in such markets is computation of the price of electricity provided by power generation companies.

A widely used paradigm for representing producers' behavior in oligopolistic electricity markets is the so called Nash-Cournot model, dealing with the analysis of market equilibria (see [2,5,10,12,16,17]). The model of [17] is applied to the New Zealand's electricity market and makes use of dual dynamic programming. In [12], the market equilibrium is stated using optimality conditions in the form of a mixed complementarity problem (MCP), which can be solved using MCP methods (for realistically sized problems). Cournot and supply function equilibria are numerically compared in [2], considering transmission constraints. In [16], some theoretical results concerning the Cournot model applied to short-term electricity markets are presented. Bilevel optimization is proposed in [8,15] to model a hydrothermal coordination problem in the presence of pumped storage units. The numerical resolution makes use of dynamic programming and some heuristics, and an application to a real case of the Argentinian electricity market is presented. An extended problem formulated as a nonmonotone variational inequality is studied in [5], including nonsmooth demand functions (e.g., piecewise concave). Although this formulation gives a more realistic model, its numerical resolution requires introduction of certain simplifications.

In [10], a model for short-term optimal scheduling of hydraulic and thermal electricity generation units based on Nash-Cournot equilibrium theory is presented. In this model, the hydraulic units have the ability of pumping water back in order to reuse it (an approach similar to [8]), leading to non-differentiability in the problem formulation.

In this work, the general approach of [10] is followed, but transmission constraints are introduced giving more realism to the model. The resulting Nash-Cournot equilibrium conditions are formulated as a pair of variational inequalities coupled by a linear constraint. The problem is then stated as a variational inclusion with a maximal monotone operator. It turns out that its structure is suitable for applying the decomposition method developed in [7] (which itself is an extension of [18]).

The rest of the paper is organized as follows. In section 2, the electricity production market and the Nash-Cournot models are described, and the notation is introduced. Section 3 is organized in two subsections. In the first, the analytical model of the problem is formulated as a variational inclusion and it is shown that it can be tackled by the decomposition method of [7] (whose description is deferred to the Appendix). The second subsection deals with the implementational issues. Section 4 presents numerical results for a small-size academic example and for a medium-size real-life system. Finally, in the last section some conclusions are stated.

2 Description of the Problem

We consider an oligopolistic electricity market with the following characteristics. The electricity production system is composed by thermal and pumped-storage hydroelectric power units, interconnected by a bounded capacity net. In thermal units, the turbines are driven by a high pressure fluid flow produced by the burn of fossil fuel, with the consequent pollutant emissions. In contrast, in a hydraulic unit the falling water rotates the turbine blades, but the availability of water depends on weather and is not guaranteed.

Pumped storage plants allow a more rational use of the hydraulic resources of a country, storing water when the electricity demand is low (off-peak hours) and using it to supply power when demand is high (peak hours). Specifically, two reservoirs at different levels, connected by a penstock and a reversible turbine, allow to generate electricity as a conventional hydro-power plant in peak hours, and to pump water back in off-peak hours, guaranteeing more water to produce energy during periods of peak consumption.

The key issue when scheduling hydroelectric plants is to design the best strategy to manage the available water. Therefore, it is necessary to take into account the total system generation during the present period, and the possibility to store water to spend it in the future. Thus, the problem is coupled through different periods, with the aim to maximize the sum of the future and the current profits.

In this setting, each operator seeks to maximize its own profit arriving to a Nash equilibrium point. To set the analytical model, the benefits of the thermal and the hydroelectric unit operators are defined. At each time period t, the operators will be paid a market price $p_t > 0$ for the quantity of energy produced. The Nash equilibrium is studied considering T periods. We shall call y_{jt} the hydroelectric generation (or consumption, if this quantity is negative) at hydroelectric unit j for time period t. The production at thermal unit i for the time period t is denoted by x_{it}. The system consists of $\mathcal{I}$ thermal plants owned by M

companies, and $\mathcal{J}$ pumped storage hydroelectric plants owned by N companies. Thus, there are M sets $\mathcal{C}_m^{Th}$ of indices representing $\mathcal{I}_m$ thermal plants each, and N sets $\mathcal{C}_n^H$ of indices representing $\mathcal{J}_m$ hydroelectric units each, with $\sum_{m=1}^M \mathcal{I}_m = \mathcal{I}$ and $\sum_{n=1}^N \mathcal{J}_n = \mathcal{J}$. So the variables for the thermal and hydroelectric production in T periods are $x = (x_{it}) \in \mathbb{R}^{\mathcal{I}T}$ and $y = (y_{jt}) \in \mathbb{R}^{\mathcal{J}T}$, respectively. The hydroelectric benefit for company n, assuming no production costs, is given by

$$Ben_n^H = \sum_{j \in \mathcal{C}_n^H} \sum_{t=1}^{T} f(y_{jt}) p_t, \tag{2.1}$$

where

$$f(s) = \begin{cases} s, & \text{if} \quad s \geq 0, \\ \alpha s, & \text{if} \quad s < 0, \end{cases}$$

is used to represent the difference between pumping ($s < 0$) and generating ($s > 0$). The efficiency coefficient $\alpha > 1$ indicates that the energy used to pump water is higher than the energy generated by the same volume of water. Note that the function f introduces a non-smoothness at zero. The main question is to determine the optimal total volume of water to be spent in the planning horizon. In this work, we assume that a known fixed volume of water is available for using in the planning horizon (obviously, less than the total volume of water in the reservoir) as a result of long-term programming that takes into account other modeling aspects (uncertainty in weather, demand, etc.), see [21] for a detailed discussion of this issue. The water available to be used by unit j during the planning horizon is represented by the total power y_j^{Tot} that the plant can generate with it. So the production schedule for each unit must satisfy the condition

$$\sum_t y_{jt} = y_j^{Tot}. \tag{2.2}$$

The thermal benefit for company m is given by

$$Ben_m^{Th} = \sum_{i \in \mathcal{C}_m^{Th}} \sum_{t=1}^{T} (x_{it} p_t - c_i^{Th}(x_{it})), \tag{2.3}$$

where c_i^{Th} is the thermal production cost for unit i.

To reduce and simplify the thermal scheduling problem, any on-off restrictions in the thermal unit operation are ignored.

The electricity system net is composed by buses (nodes) and lines (arcs). Some pairs of buses, say b and k, are linked by a line $\ell = (b, k)$. The power derived by bus b to line ℓ for time period t is denoted by w_{bkt}, with $w_{bkt} = -w_{kbt}$. The electricity net capacity constraints are given by

$$|w_{bkt}| \leq w_{bk}^{Cap}, \qquad \forall b, k, t \tag{2.4}$$

and

$$\sum_{i \in \mathcal{S}_b^{Th}} x_{it} + \sum_{j \in \mathcal{S}_b^H} y_{jt} - d_{bt} = \sum_{k \in \mathcal{S}_b^B} w_{bkt}, \qquad \forall b, t, \tag{2.5}$$

where $\mathcal{S}_b^{Th}$, $\mathcal{S}_b^H$, $\mathcal{S}_b^B$ are respectively the sets of thermal plants, hydraulic plants and buses linked with bus b, and d_{bt} is the demand on bus b at time period t. These demands are related with the market price p through the demand functions (assumed affine)

$$d_{bt}(p) = D_{bt} - a_{bt} p, \tag{2.6}$$

whose coefficients D_{bt} and a_{bt} are obtained given the value of the elasticity around a demand-price anchor point. Therefore, the market price at time period t can be obtained by the inverse demand function (IDF)

$$p(d_{bt}) = \frac{1}{a_{bt}} \left(D_{bt} - d_{bt}\right), \tag{2.7}$$

or summing up over b in (2.6),

$$p(d_t) = \frac{1}{a_t} \left(D_t - d_t\right), \tag{2.8}$$

where $a_t = \sum_{b=1}^{\mathcal{B}} a_{bt}$, $D_t = \sum_{b=1}^{\mathcal{B}} D_{bt}$ and $d_t = \sum_{b=1}^{\mathcal{B}} d_{bt}$ and $\mathcal{B}$ is the number of buses in the network.

Nash equilibrium means that the demand is supplied by all the players. Thus, the total production of energy at time t is composed of thermal and hydroelectric production and is equal to the sum of the bus demands:

$$\sum_{j=1}^{\mathcal{J}} y_{jt} + \sum_{i=1}^{\mathcal{I}} x_{it} = \sum_{b=1}^{\mathcal{B}} d_{bt}. \tag{2.9}$$

Therefore, using (2.9), the market price becomes

$$p_t = \frac{1}{a_t} \left(D_t - \sum_{j=1}^{\mathcal{J}} y_{jt} - \sum_{i=1}^{\mathcal{I}} x_{it} \right), \tag{2.10}$$

where naturally $p_t > 0$, and the benefits for each type of plants would depend on their own production and the productions of the other plants. Also, the constraints (2.5), for all b, t, take the form

$$\sum_{i \in \mathcal{S}_b^{Th}} x_{it} + \sum_{j \in \mathcal{S}_b^{H}} y_{jt} - D_{bt} + \frac{a_{bt}}{a_t} \left(D_t - \sum_{j=1}^{\mathcal{J}} y_{jt} - \sum_{i=1}^{\mathcal{I}} x_{it} \right) = \sum_{k \in \mathcal{S}_b^{B}} w_{bkt}. \tag{2.11}$$

Therefore, the constraints for each hydroelectric unit are given by the production bounds y_j^{Low}, y_j^{Up} and by (2.2). The n-th company constraints set is given by

$$\mathcal{K}_n^H = \{y_j \in \mathbb{R}^{\mathcal{J}_n T} : y_j^{Low} \le y_{jt} \le y_j^{Up}, t = 1, .., T, \sum_t y_{jt} = y_j^{Tot}, j \in \mathcal{C}_n^H\}. \tag{2.12}$$

For each thermal unit only the restrictions given by production bounds x_i^{Low}, x_i^{Up} are considered, so the constraints set for company m is the box

$$\mathcal{K}_m^{Th} = \{x_m \in \mathbb{R}^{\mathcal{I}_m T} : x_i^{Low} \le x_{it} \le x_i^{Up}, t = 1, .., T, i \in \mathcal{C}_m^{Th}\}.$$

The conditions for the Nash equilibrium considering the production vector (x^*, y^*) are

$$Ben_m^{Th}(x^*, y^*) = \max_{x_m \in \mathcal{K}_m^{Th}} Ben_m^{Th}(x_m, x^*_{/m}, y^*), \qquad m = 1, ..., M, \tag{2.13}$$

and, simultaneously,

$$Ben_n^H(x^*, y^*) = \max_{y_n \in \mathcal{K}_n^H} Ben_n^H(x^*, y_n, y_{/n}^*), \qquad n = 1, ..., N, \tag{2.14}$$

where $x_m = (x_{it})_{i \in \mathcal{C}_m^{Th}}$, $x_{/m} = (x_{it})_{i \notin \mathcal{C}_m^{Th}}$, $y_n = (y_{jt})_{j \in \mathcal{C}_n^H}$ and $y_{/n} = (y_{jt})_{j \notin \mathcal{C}_n^H}$.

The Net Constrained Nash-Cournot Equilibrium Problem (NCNCEP) is to determine x, w and y satisfying conditions (2.13) and (2.14), together with the linear constraint (2.11).

The unconstrained Nash-Cournot Equilibrium Problem (NCEP), i.e., without the net constraints (2.11), was previously studied in [10]. There, in order to deal with the non-smoothness of the hydroelectric benefit functions, the variable y is split in the form $y = y^+ - y^-$ of positive and negative parts, where

$$(y^+)_{jt} = \begin{cases} y_{jt}, & \text{if} \quad y_{jt} \geq 0, \\ 0, & \text{if} \quad y_{jt} < 0, \end{cases} \quad (y^-)_{jt} = \begin{cases} 0, & \text{if} \quad y_{jt} \geq 0, \\ -y_{jt}, & \text{if} \quad y_{jt} < 0, \end{cases} \tag{2.15}$$

for $j = 1, .., \mathcal{J}$ and $t = 1, .., T$. Then, defining $z = \left(y^{+\top}, y^{-\top}\right)$, it turns out that $y = Rz$ where $R = (I_{\mathcal{J}T}, -I_{\mathcal{J}T})$, and the hydraulic benefits turn out to be quadratic functions of the variables $z_n \in \bar{\mathcal{K}}_n^H$. The new constraints set derived from (2.12) with the formulation (2.15) involves a complementarity constraint. On the other hand, the thermal costs are assumed quadratic. So, from (2.3) and (2.10), the thermal benefits are quadratic functions in the variables x_n. NCEP can be stated as the following variational inequality: Find $(x^*, z^*) \in \mathcal{K}^{Th} \times \bar{\mathcal{K}}^H$ such that

$$\Psi \begin{pmatrix} x^* \\ z^* \end{pmatrix}^\top \begin{pmatrix} x - x^* \\ z - z^* \end{pmatrix} \geq 0, \qquad \forall (x, z) \in \mathcal{K}^{Th} \times \bar{\mathcal{K}}^H, \tag{2.16}$$

with

$$\mathcal{K}^{Th} = \prod_{m=1}^{M} \mathcal{K}_m^{Th}, \quad \bar{\mathcal{K}}^H = \prod_{n=1}^{N} \bar{\mathcal{K}}_n^H,$$

and

$$\Psi \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} \left(\nabla_{x_m} Ben_m^{Th}(x_m, x_{/m}, z)\right)_{m=1,..,M} \\ \left(\nabla_{z_n} Ben_m^H(x, z_n, z_{/n})\right)_{n=1,..,N} \end{pmatrix} = \begin{pmatrix} M^{Th} x + \Gamma z + \gamma \\ M^H z + \Theta x + \theta \end{pmatrix},$$

for adequate matrices $M^{Th}, M^H, \Gamma, \Theta$ and vectors γ, θ. A precise description of the matrices and vectors involved can be found in [10]. In particular, M^{Th} is symmetric but M^H is not. A key issue is that the matrix

$$\begin{pmatrix} M^{Th} & \Gamma \\ \Theta & M^H \end{pmatrix} \tag{2.17}$$

is positive semidefinite, which is not difficult to prove from the definitions of the matrices involved. In the following section, the formulation (2.16) is used as a starting point for the analysis of NCNCEP.

3 Net-Constrained Nash-Cournot Equilibrium Problem

In this section, the analytic model for NCNCEP is introduced, and it is shown that it can be solved by the decomposition scheme derived from the Variable Metric Hybrid Proximal

Decomposition Method (VMHPDM). We refer to [7] for a detailed description of this method in the general form. A brief (simplified) outline is given in the Appendix. In short, we need to show that the problem can be cast as a variational inclusion $0 \in T(z)$, where the operator T has following structure:

$$T = F \times [G + H], \tag{3.1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is maximal monotone, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is maximal monotone.

3.1 The Analityc Model

In NCNCEP the net constraints (2.11) depend on the auxiliary box-constrained variable w. Let $\mathcal{C}^N$ be the index set of the lines $\ell = (b, k)$, with $\mathcal{L}$ elements, where it is assumed that line (b, k) is the same as line (k, b). So only one of these pairs is considered in the definition of ℓ. Since $w_{bkt} = -w_{kbt}\,\forall b, k, t$, it is natural to use only one variable w_ℓ, and change the sings accordingly in (2.11). Then the constraint set for w is given by

$$\mathcal{K}^N = \{w \in \mathbb{R}^{\mathcal{L}T} : -w_\ell^{Cap} \le w_{\ell t} \le w_\ell^{Cap}, \ell \in \mathcal{C}^N, t = 1, .., T\}.$$

We next write the net constraints (2.11) in matrix form. For this, the following incidence matrices are defined:

$$A_0 \in \mathbb{R}^{\mathcal{B}\times\mathcal{L}} \quad \text{s.t.} \quad (A_0)_{b,\ell} = \begin{cases} -1, & \text{if } \ell = (b,k) \text{ for some bus } k, \\ 1, & \text{if } \ell = (k,b) \text{ for some bus } k, \\ 0, & \text{otherwise,} \end{cases}$$

$$B_0 \in \mathbb{R}^{\mathcal{B}\times\mathcal{I}} \quad \text{s.t.} \quad (B_0)_{b,i} = \begin{cases} 1, & \text{if } i \in \mathcal{S}_b^{Th}, \\ 0, & \text{if } i \notin \mathcal{S}_b^{Th}, \end{cases}$$

$$C_0 \in \mathbb{R}^{\mathcal{B}\times\mathcal{J}} \quad \text{s.t.} \quad (C_0)_{b,j} = \begin{cases} 1, & \text{if } j \in \mathcal{S}_b^{H}, \\ 0, & \text{if } j \notin \mathcal{S}_b^{H}. \end{cases}$$

Also, for $t = 1, \ldots, T$, we define

$$v_t \in \mathbb{R}^{\mathcal{B}} \quad \text{s.t.} \quad (v_t)_b = D_{bt} - \frac{a_{bt}}{a_t} D_t,$$

$$\begin{array}{llll} B_t \in \mathbb{R}^{\mathcal{B}\times\mathcal{I}} & \text{s.t.} & (B_t)_{b,i} = & a_{bt}/a_t, \forall i = 1, \ldots, \mathcal{I}, \\ C_t \in \mathbb{R}^{\mathcal{B}\times\mathcal{J}} & \text{s.t.} & (C_t)_{b,j} = & a_{bt}/a_t, \forall i = 1, \ldots, \mathcal{J}. \end{array}$$

With these definitions, the linear constraints (2.11) become

$$Aw + Bx + CRz - v = 0, \tag{3.2}$$

where $v = (v_1^\top, v_2^\top, \ldots, v_T^\top)^\top \in \mathbb{R}^{\mathcal{B}T}$ and

$$A \in \mathbb{R}^{\mathcal{B}T\times\mathcal{L}T} \quad \text{s.t.} \quad A = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & A_0 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix},$$

$$B \in \mathbb{R}^{\mathcal{B}T\times\mathcal{I}T} \quad \text{s.t.} \quad B = \begin{pmatrix} B_0 - B_1 & 0 & 0 & 0 \\ 0 & B_0 - B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_0 - B_T \end{pmatrix},$$

$$C \in \mathbb{R}^{\mathcal{B}T\times\mathcal{J}T} \quad \text{s.t.} \quad C = \begin{pmatrix} C_0 - C_1 & 0 & 0 & 0 \\ 0 & C_0 - C_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & C_0 - C_T \end{pmatrix}.$$

Then a solution of NCNCEP is a point (w^*, x^*, z^*) satisfying (2.16) and (3.2). In particular, it must be a feasible point for the set

$$\bar{\mathcal{K}} = \left\{(w, x, z) \in \mathcal{K}^N \times \mathcal{K}^{Th} \times \bar{\mathcal{K}}^H : Aw + Bx + CRz - v = 0\right\}. \tag{3.3}$$

It is useful to state the hydraulic constraints set in the following way. Let $\mathcal{K}^H$ be the box

$$\mathcal{K}^H = \left\{z \in \mathbb{R}^{2\mathcal{J}T} : 0 \le z \le Z^{Up}\right\}, \tag{3.4}$$

where $Z^{Up} = ({y^{Up}}^\top, \cdots {y^{Up}}^\top, -{y^{Low}}^\top, \cdots, -{y^{Low}}^\top)^\top \in \mathbb{R}^{2\mathcal{J}T}$, and define the matrices

$$E = \begin{pmatrix} 0 & I_{\mathcal{J}T} \\ I_{\mathcal{J}T} & 0 \end{pmatrix} \in \mathbb{R}^{2\mathcal{J}T\times 2\mathcal{J}T}, \quad S = (I_{\mathcal{J}}, \cdots, I_{\mathcal{J}}) \in \mathbb{R}^{\mathcal{J}\times\mathcal{J}T}.$$

Then the hydraulic constraints set is given by

$$\bar{\mathcal{K}}^H = \left\{z \in \mathcal{K}^H : SRz = y^{Tot}, z^\top E z = 0\right\}. \tag{3.5}$$

We next show that if we solve NCNCEP on the set with the complementarity constraint in (3.5) ignored, then any such solution satisfies this constraint automatically. Hence, it can be simply removed from the problem formulation. The result essentially follows adapting [10, Proposition 1] to the current setting. In order to take care of some details and for the sake of completeness, we include a streamlined proof.

Proposition 3.1. *Let (x^*, w^*, z^*) be a solution of NCNCEP with the relaxed constraints set*

$$\hat{\mathcal{K}} = \left\{(w, x, z) \in \mathcal{K}^N \times \mathcal{K}^{Th} \times \mathcal{K}^H : SRz = y^{Tot}, Aw + Bx + CRz - v = 0\right\}, \tag{3.6}$$

i.e., $(x^*, w^*, z^*) \in \hat{\mathcal{K}}$ satisfies

$$Ben_n^H(x^*, z^*) = \max_{z_n \in \mathcal{K}_n^H} Ben_n^H(x^*, z_n, z^*_{/n}), \qquad n = 1, ..., N, \tag{3.7}$$

$$Ben_m^{Th}(x^*, z^*) = \max_{x_m \in \mathcal{K}_m^{Th}} Ben_m^{Th}(x_m, x^*_{/m}, z^*), \qquad m = 1, ..., M, \tag{3.8}$$

and

$$Aw^* + Bx^* + CRz^* - v = 0. \tag{3.9}$$

Then it holds that ${z^*}^\top E z^* = 0$.

Proof. Suppose that (x^*, w^*, z^*) satisfies (3.7), (3.8) and (3.9), but ${z^*}^\top E z^* \neq 0$. Then, since $z^* \geq 0$, there exist indices j_0 and t_0 such that

$$\begin{aligned} z^*_{j_0 t_0} &= (y^*)^+_{j_0 t_0} > 0, \\ z^*_{j_0 t_0 + \mathcal{J}T} &= (y^*)^-_{j_0 t_0} > 0. \end{aligned}$$

Define the new point $\bar{z}$ by

$$\begin{aligned} \bar{z}_{jt} &= z^*_{jt}, \text{ for } jt \neq j_0 t_0 \text{ and } jt \neq j_0 t_0 + \mathcal{J}T, \\ \bar{z}_{j_0 t_0} &= \begin{cases} z^*_{j_0 t_0} - z^*_{j_0 t_0 + \mathcal{J}T}, & \text{if} \quad z^*_{j_0 t_0} \geq z^*_{j_0 t_0 + \mathcal{J}T}, \\ 0, & \text{if} \quad z^*_{j_0 t_0} < z^*_{j_0 t_0 + \mathcal{J}T}, \end{cases} \\ \bar{z}_{j_0 t_0 + \mathcal{J}T} &= \begin{cases} 0, & \text{if} \quad z^*_{j_0 t_0} \geq z^*_{j_0 t_0 + \mathcal{J}T}, \\ z^*_{j_0 t_0 + \mathcal{J}T} - z^*_{j_0 t_0}, & \text{if} \quad z^*_{j_0 t_0} < z^*_{j_0 t_0 + \mathcal{J}T}. \end{cases} \end{aligned}$$

Since $\bar{z}_{jt} - \bar{z}_{jt+\mathcal{J}T} = z^*_{jt} - z^*_{jt+\mathcal{J}T}$ for all (j, t), it is straightforward that $\bar{z}$ satisfies the hydraulic constraints and also that $CR\bar{z} = CRz^*$, which implies that $Aw^* + Bx^* + CR\bar{z} - v = 0$.

Moreover, for all $t = 1, \ldots, T$, the prices p_t satisfy $p_t(x^*, \bar{z}) = p_t(x^*, z^*)$ and hence, for all $m \in \{1, \ldots, M\}$ and all $n \in \{n : 1 \leq n \leq N, j_0 \notin \mathcal{C}_n^H\}$, the benefits satisfy

$$\begin{aligned} Ben_m^{Th}(x^*_m, x^*_{/m}, \bar{z}) &= Ben_m^{Th}(x^*_m, x^*_{/m}, z^*), \\ Ben_n^H(x^*, \bar{z}_n, \bar{z}_{/n}) &= Ben_n^H(x^*, z^*_n, z^*_{/n}). \end{aligned}$$

If $z^*_{j_0 t_0} \geq z^*_{j_0 t_0 + \mathcal{J}T}$ then using that $\alpha_{j_0} > 1$ and $p_{t_0} > 0$, we obtain that

$$\begin{aligned} f(\bar{y}_{j_0 t_0}) p_{t_0} &= (\bar{z}_{j_0 t_0} - \alpha_{j_0} \bar{z}_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = (z^*_{j_0 t_0} - z^*_{j_0 t_0 + \mathcal{J}T}) p_{t_0} \\ &> (z^*_{j_0 t_0} - \alpha_{j_0} z^*_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = f(y^*_{j_0 t_0}) p_{t_0}. \end{aligned}$$

On the other hand, if $z^*_{j_0 t_0} < z^*_{j_0 t_0 + \mathcal{J}T}$ then

$$\begin{aligned} f(\bar{y}_{j_0 t_0}) p_{t_0} &= (\bar{z}_{j_0 t_0} - \alpha_{j_0} \bar{z}_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = \alpha_{j_0} (z^*_{j_0 t_0} - z^*_{j_0 t_0 + \mathcal{J}T}) p_{t_0} \\ &> (z^*_{j_0 t_0} - \alpha_{j_0} z^*_{j_0 t_0 + \mathcal{J}T}) p_{t_0} = f(y^*_{j_0 t_0}) p_{t_0}. \end{aligned}$$

Therefore, for n_0 such that $j_0 \in \mathcal{C}^H_{n_0}$, for the benefit of company n_0 it holds that

$$Ben^H_{n_0}(x^*, \bar{z}_{n_0}, \bar{z}_{/n_0}) > Ben^H_m(x^*, z^*_{n_0}, z^*_{/n_0}),$$

in contradicton with (3.7). This establishes that ${z^*}^\top E z^* = 0$ □

Now, taking into account (2.16) with the subsequent definitions, (3.2), Proposition 3.1 and definition (3.6), NCNCEP can be stated as the following variational inequality: Find $(w^*, x^*, z^*) \in \hat{\mathcal{K}}$ such that

$$\begin{pmatrix} 0 \\ M^{Th} x^* + \Gamma z^* + \gamma \\ \Theta x^* + M^H z^* + \theta \end{pmatrix}^\top \begin{pmatrix} w - w^* \\ x - x^* \\ z - z^* \end{pmatrix} \geq 0, \qquad \forall (w, x, z) \in \hat{\mathcal{K}}. \tag{3.10}$$

Next, associating a Lagrange multiplier to the equality constraints in $\hat{\mathcal{K}}$, (3.10) becomes equivalent to the following variational inequality in the primal-dual space (see, e.g., [4, Proposition 1.3.4]): Find $(w^*, x^*, z^*, \mu^*) \in \mathcal{K} = \mathcal{K}^N \times \mathcal{K}^{Th} \times \mathcal{K}^H \times \mathbb{R}^{\mathcal{B}T+\mathcal{J}}$ such that

$$\begin{pmatrix} \bar{A}^\top \mu^* \\ M^{Th}x^* + \Gamma z^* + \gamma + B^\top \mu^* \\ \Theta x^* + M^H z^* + \theta + \bar{C}^\top \mu^* \\ \bar{v} - \bar{A}w^* - \bar{B}x^* - \bar{C}z^* \end{pmatrix}^\top \begin{pmatrix} w - w^* \\ x - x^* \\ z - z^* \\ \mu - \mu^* \end{pmatrix} \geq 0, \qquad \forall (w, x, z, \mu) \in \mathcal{K}, \tag{3.11}$$

where

$$\bar{A} = \begin{bmatrix} A \\ 0 \end{bmatrix} \in \mathbb{R}^{(\mathcal{B}T+\mathcal{J})\times \mathcal{L}T}, \qquad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \in \mathbb{R}^{(\mathcal{B}T+\mathcal{J})\times \mathcal{I}T}$$

and

$$\bar{C} = \begin{bmatrix} CR \\ SR \end{bmatrix} \in \mathbb{R}^{(\mathcal{B}T+\mathcal{J})\times 2\mathcal{J}T}, \quad \text{and} \quad \bar{v} = \begin{bmatrix} v \\ y^{Tot} \end{bmatrix} \in \mathbb{R}^{\mathcal{B}T+\mathcal{J}}.$$

As is well known, (3.11) is equivalent to the variational inclusion

$$0 \in (\Phi + N_{\mathcal{K}})(w^*, x^*, z^*, \mu^*), \tag{3.12}$$

where

$$\Phi(w, x, z, \mu) = \begin{pmatrix} \bar{A}^\top \mu \\ M^{Th}x + \Gamma z + \gamma + \bar{B}^\top \mu \\ \Theta x + M^H z + \theta + \bar{C}^\top \mu \\ \bar{v} - \bar{A}w - \bar{B}x - \bar{C}z \end{pmatrix}$$

and $N_{\mathcal{K}}(w, x, z, \mu)$ is the normal cone to the closed convex set $\mathcal{K}$ at the point (w, x, z, μ).

Note that the sum of a normal cone to a closed convex set with a monotone (single-valued) continuous function is a maximal monotone operator. It is also easy to see that this operator $\Phi + N_{\mathcal{K}}$ has the separable structure in (3.1). In particular, taking $u = (w, x)$ and $v = (z, \mu)$, we have that

$$(\Phi + N_{\mathcal{K}})(w, x, z, \mu) = F(w, x, z, \mu) \times (G(w, x, z, \mu) + H(z, \mu)),$$

where $F : \mathbb{R}^{(\mathcal{L}+\mathcal{I})T} \times \mathbb{R}^{(2\mathcal{J}+\mathcal{B})T+\mathcal{J}} \rightrightarrows \mathbb{R}^{(\mathcal{L}+\mathcal{I})T}$ is the set-valued operator given by

$$F(w, x, z, \mu) = \begin{pmatrix} \bar{A}^\top \mu + N_{\mathcal{K}^N}(w) \\ M^{Th}x + \Gamma z + \gamma + \bar{B}^\top \mu + N_{\mathcal{K}^{Th}}(x) \end{pmatrix}, \tag{3.13}$$

$G : \mathbb{R}^{(\mathcal{L}+\mathcal{I})T} \times \mathbb{R}^{(2\mathcal{J}+\mathcal{B})T+\mathcal{J}} \to \mathbb{R}^{(2\mathcal{J}+\mathcal{B})T+\mathcal{J}}$ is the continuous function

$$G(w, x, z, \mu) = \begin{pmatrix} \Theta x + M^H z + \theta + \bar{C}^\top \mu \\ \bar{v} - \bar{A}w - \bar{B}x - \bar{C}z \end{pmatrix}, \tag{3.14}$$

and $H : \mathbb{R}^{(2\mathcal{J}+\mathcal{B})T+\mathcal{J}} \rightrightarrows \mathbb{R}^{(2\mathcal{J}+\mathcal{B})T+\mathcal{J}}$ is the maximal monotone operator

$$H(z, \mu) = \begin{pmatrix} N_{\mathcal{K}^H}(z) \\ 0 \end{pmatrix}. \tag{3.15}$$

Convergence of the decomposition scheme requires that certain assumptions are met; see the Appendix. In the present context, assumptions A1, A2 and A4 therein are trivially satisfied. Assumption A3 holds because the matrix (2.17) is positive semidefinite, and because sums of maximal monotone operators are maximal monotone provided a constraint qualification holds (see [13, 14]).

3.2 The Algorithm

We now state the specific decomposition algorithm for our application, and justify that it is well defined and is indeed a special case of VMHPDM (in particular, convergence and rate of convergence then follow from [7]). Details we refer to the Appendix; they can be skipped at first reading.

Algorithm 3.2. (Decomposition for NCNCEP)
`Initialization:` Choose $(w^0, x^0, z^0, \mu^0) \in \mathbb{R}^{\mathcal{L}T} \times \mathbb{R}^{\mathcal{I}T} \times \mathbb{R}^{2\mathcal{J}T} \times \mathbb{R}^{\mathcal{B}T+\mathcal{J}}$. Set $k := 0$.

`Forward-Backward Splitting Step:` For $c_k > 0$, compute

$$\hat{z}^k = \mathrm{Proj}_{\mathcal{K}^H} \left\{ z^k - c_k \left(\Theta x^k + M^H z^k + \theta + \bar{C}^\top \mu^k \right) \right\}, \tag{3.16}$$

$$\hat{\mu}^k = \mu^k - c_k(\bar{v} - \bar{A} w^k - \bar{B} x^k - \bar{C} z^k). \tag{3.17}$$

`Proximal Step:` For appropriately chosen $\beta > 0$ and a symmetric positive definite matrix U (see (3.27) below), compute

$$\hat{w}^k = \mathrm{Proj}_{\mathcal{K}^N} \left\{ w^k - c_k \bar{A}^\top \hat{\mu}^k \right\}, \tag{3.18}$$

$$\hat{x}^k = \mathrm{Proj}_{\mathcal{K}^{Th}} \left\{ \frac{1}{\beta} \left(U x^k - (\Gamma \hat{z}^k + \gamma + \bar{B}^\top \hat{\mu}^k) \right) \right\}. \tag{3.19}$$

`Approximation condition test:` Choose the error tolerance parameter $\sigma_k \in (0, 1)$. If the inequality

$$\|s^k\|^2 \le \sigma_k^2 (\|\hat{w}^k - w^k\|^2 + c_k \|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2 + \|\hat{\mu}^k - \mu^k\|^2), \tag{3.20}$$

with

$$s^k = (s_z^k, s_\mu^k) = c_k \left(G(\hat{w}^k, \hat{x}^k, \hat{z}^k, \hat{\mu}^k) - G(w^k, x^k, z^k, \mu^k) \right) \tag{3.21}$$

is not satisfied, decrease c_k and go to `Forward-Backward Splitting Step`. Otherwise proceed.
`Iterates Update:` Stop if $\hat{w}^k = w^k$, $\hat{x}^k = x^k$, $\hat{z}^k = z^k$ and $\hat{\mu}^k = \mu^k$. Otherwise, define

$$\begin{aligned} w^{k+1} &= \hat{w}^k \\ x^{k+1} &= \hat{x}^k \\ z^{k+1} &= \hat{z}^k - s_z^k \\ \mu^{k+1} &= \hat{\mu}^k - s_\mu^k. \end{aligned} \tag{3.22}$$

Set $k := k + 1$ and go to `Forward-Backward Splitting Step`.

The next proposition shows that steps (3.16)-(3.17) and (3.18)-(3.19) in Algorithm 3.2 are indeed forward-backward and proximal steps in the general VMHPDM framework, respectively.

Proposition 3.3. *It holds that the relations* (3.16) *and* (3.17) *are equivalent to the forward-backward splitting step*

$$(\hat{z}^k, \hat{\mu}^k) = (I + c_k H(\cdot, \cdot))^{-1} \left(I - c_k G(w^k, x^k, z^k, \mu^k) \right). \tag{3.23}$$

It holds that for an appropriate $\beta > 0$ and a symmetric positive definite matrix U, the relations (3.18) *and* (3.19) *are equivalent to the exact proximal step consisting in solving the inclusion*

$$0 \in F(\hat{w}^k, \hat{x}^k, \hat{z}^k, \hat{\mu}^k) + \frac{1}{c_k} \begin{pmatrix} I & 0 \\ 0 & c_k U \end{pmatrix} \begin{pmatrix} \hat{w}^k - w^k \\ \hat{x}^k - x^k \end{pmatrix}. \tag{3.24}$$

Proof. From (3.14) and (3.15), the relation (3.23) gives

$$\begin{array}{rcl} \hat z^k + N_{\mathcal{K}^H}(\hat z^k) & \ni & z^k - c_k\left(\Theta x^k + M^H z^k + \theta + \bar C^\top \mu^k\right), \\ \hat\mu^k & = & \mu^k - c_k(\bar v - \bar A w^k - \bar B x^k - \bar C z^k). \end{array}$$

The equality above is exactly (3.17). Also, recalling the basic property of the projection operator (namely that $\hat s = \mathrm{Proj}_{\mathcal{K}^H}\{s\}$ if and only if $s - \hat s \in N_{\mathcal{K}^H}(\hat s)$), the inclusion above means (3.16).

Using (3.13), the inclusion (3.24) is equivalent to

$$0 \in \frac{1}{c_k}\hat w^k + \bar A^\top \hat\mu^k - \frac{1}{c_k} w^k + N_{\mathcal{K}^N}(\hat w^k), \quad (3.25)$$

$$0 \in (U + M^{Th})\hat x^k + \Gamma \hat z^k + \gamma + \bar B^\top \hat\mu^k - U x^k + N_{\mathcal{K}^{Th}}(\hat x^k). \quad (3.26)$$

Let

$$U = \beta I_{\mathcal{I}T} - M^{Th} \quad \text{with} \quad \beta > \max_{k=1,\ldots,\mathcal{I}T}\left\{\sum_{j=1}^{\mathcal{I}T} |(M^{Th})_{kj}|\right\}. \quad (3.27)$$

Note that according to (3.27), U is symmetric and diagonally dominant. Hence, it is positive definite. Also,

$$U + M^{Th} = \beta I_{\mathcal{I}T}.$$

Since for any cone K its holds that $K = tK$ for any $t > 0$, multiplying by c_k in (3.25) and dividing by β in (3.26), the proximal step takes the form

$$\begin{array}{rcl} w^k - c_k \bar A^\top \hat\mu^k & \in & \hat w^k + N_{\mathcal{K}^N}(\hat w^k), \\ \frac{1}{\beta}\left(U x^k - (\Gamma \hat z^k + \gamma + \bar B^\top \hat\mu^k)\right) & \in & \hat x^k + N_{\mathcal{K}^{Th}}(\hat x^k). \end{array}$$

Using again the same basic property of the projection as above, these two relations give (3.18) and (3.19), respectively. □

Remark 3.4. As can be seen in the Appendix, steps (3.23) and (3.24) are special cases of the steps (6.2) and (6.3) therein, respectively, for $u = (w, x)$, $v = (z, \mu)$, $Q_k = I$ and $P_k^{-1} = \begin{pmatrix} I & 0 \\ 0 & c_k U \end{pmatrix}$.

Remark 3.5. Note that since the sets $\mathcal{K}^H$, $\mathcal{K}^N$ and $\mathcal{K}^{Th}$ are defined by box constraints, the projections (3.16), (3.18) and (3.19) can be computed explicitly by the formulas

$$\hat z^k = \min\left\{\max\{z^k - c_k\left(\Theta x^k + M^H z^k + \theta + \bar C^\top \mu^k\right), 0\}, Z^{Up}\right\}, \quad (3.28)$$

$$\hat w^k = \min\left\{\max\{w^k - c_k \bar A^\top \hat\mu^k, -W^{Cap}\}, W^{Cap}\right\} \quad (3.29)$$

and

$$\hat x^k = \min\left\{\max\{\frac{1}{\beta}\left(U x^k - (\Gamma \hat z^k + \gamma + \bar B^\top \hat\mu^k)\right), X^{Low}\}, X^{Up}\right\}, \quad (3.30)$$

where

$$W^{Cap} = ({w^{Cap}}^\top, \ldots, {w^{Cap}}^\top)^\top \in \mathbb{R}^{\mathcal{L}T}$$

and

$$X^{Low} = ({x^{Low}}^\top, \ldots, {x^{Low}}^\top)^\top, \; X^{Up} = ({x^{Up}}^\top, \ldots, {x^{Up}}^\top)^\top \in \mathbb{R}^{\mathcal{I}T}.$$

This justifies solving the steps (6.2) and (6.3) exactly in this context.

Next we show that the approximation condition (3.20) can be satisfied. The proof is analogous to the proof of [10, Proposition 3]; we include it for completeness.

Proposition 3.6. *With the choices of β and U as in* (3.27)*, there exists $\bar{c} > 0$ such that* (3.20) *holds if the operations in* (3.17)*,* (3.16)*,* (3.18) *and* (3.19) *are performed with any $c_k \in (0, \bar{c})$.*

Proof. Using (3.14) and (3.21), we see that

$$s^k = c_k \left(\begin{bmatrix} 0 \\ -\bar{A} \end{bmatrix} (\hat{w}^k - w^k) + \begin{bmatrix} \Theta \\ -\bar{B} \end{bmatrix} (\hat{x}^k - x^k) + \begin{bmatrix} M^H \\ -\bar{C} \end{bmatrix} (\hat{z}^k - z^k) + \begin{bmatrix} \bar{C}^\top \\ 0 \end{bmatrix} (\hat{\mu}^k - \mu^k) \right).$$

Let $L = \max\{\|\bar{A}^\top \bar{A}\|, \|\Theta^\top \Theta + \bar{B}^\top \bar{B}\|, \|{M^H}^\top M^H + \bar{C}^\top \bar{C}\|, \|\bar{C}\bar{C}^\top\|\}$ and $\gamma = \max(\|U^{-1}\|, 1)$. Then, using the convexity of $\|\cdot\|^2$ and assuming (without loss of generality with respect to the claim to be proved) that $c_k \le 1$, it holds that

$$\begin{aligned} \|s^k\|^2 &\le 4c_k^2 \left(\|\bar{A}(\hat{w}^k - w^k)\|^2 + \left\| \begin{bmatrix} \Theta \\ -\bar{B} \end{bmatrix} (\hat{x}^k - x^k) \right\|^2 \right. \\ &\qquad \left. + \left\| \begin{bmatrix} M^H \\ -\bar{C} \end{bmatrix} (\hat{z}^k - z^k) \right\|^2 + \|\bar{C}^\top (\hat{\mu}^k - \mu^k)\|^2 \right) \\ &\le 4c_k^2 L(\|\hat{w}^k - w^k\|^2 + \|\hat{x}^k - x^k\|^2 + \|\hat{z}^k - z^k\|^2 + \|\hat{\mu}^k - \mu^k\|^2) \qquad (3.31) \\ &\le 4c_k L(\|\hat{w}^k - w^k\|^2 + c_k\|U^{-1}\|\|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2 + \|\hat{\mu}^k - \mu^k\|^2) \\ &\le 4c_k L\gamma(\|\hat{w}^k - w^k\|^2 + c_k\|\hat{x}^k - x^k\|_U^2 + \|\hat{z}^k - z^k\|^2 + \|\hat{\mu}^k - \mu^k\|^2). \end{aligned}$$

Hence, condition (3.20) is satisfied if $c_k \le \bar{c} = \min\{\sigma_k^2/(4L\gamma), 1\}$. □

The bound $\bar{c}$ defined above is a guarantee that (3.20) can be satisfied, and thus a guarantee for theoretical convergence of Algorithm 3.2. But this bound may be too conservative. For this reason, Algorithm 3.2 allows values of c_k larger that $\bar{c}$, which are decreased only if (3.20) does not hold. If (3.20) holds at every iteration, convergence still follows regardless of the values of c_k. Moreover, for the given operator T and the choice of the matrix U, the linear local rate of convergence is obtained if the parameters c_k are bounded away from zero (see [7, Theorem 2]).

Let $\nu \in (0, 1)$ be the constant in the linear convergence estimate and let $\xi^* := (w^*, x^*, z^*, \mu^*)$ be a solution of the problem, (similarly, $\xi^k := (w^k, x^k, z^k, \mu^k)$, $\xi^{k+1} := (w^{k+1}, x^{k+1}, z^{k+1}, \mu^{k+1})$ and $\hat{\xi}^k := (\hat{w}^k, \hat{x}^k, \hat{z}^k, \hat{\mu}^k)$). Then, for all k sufficiently large, by using (3.22) and and (3.31) it holds that (see [10])

$$\|\xi^k - \xi^*\| \le \frac{1 + 2c_k\sqrt{L}}{1 - \nu} \left\| \xi^k - \hat{\xi}^k \right\|. \qquad (3.32)$$

Thus, $\|\xi^k - \hat{\xi}^k\|$ measures how far is an iterate from the solution. This justifies the stopping test used in the implementation of the algorithm, which we illustrate on some examples next.

4 Numerical Experiments

In this section, numerical results on two different systems are presented. The first test case is a small-size academic example. The second is based on a real medium-size system.

According to a short-term optimization problem, 12 and 24 hours planning horizons were taken into account, respectively.

The algorithm was coded using SciLab 5.3.3 (INRIA-ENPC, see www.scilab.org) and ran on an Intel Core I5-2430M 2.96 Ghz with 4 GB of RAM. The theoretical stopping condition of Algorithm 3.2 is $\hat{w}^k = w^k$, $\hat{x}^k = x^k$, $\hat{z}^k = z^k$ and $\hat{\mu}^k = \mu^k$. Recalling (3.32), the natural stopping criterion used in our implementation is $\|\hat{w}^k - w^k\| < \text{tol}$, $\|\hat{x}^k - x^k\| < \text{tol}$, $\|\hat{z}^k - z^k\| < \text{tol}$ and $\|\hat{\mu}^k - \mu^k\| < \text{tol}$, where tol is a tolerance parameter. By (3.32), this stopping test gives a measure of quality of the obtained iteration in terms of proximity to the exact solution. In the numerical experiences, the parameter tol was set equal to 10^{-6} (so in the case of electricity production, since the used units are megawatts, the error is about one watt).

4.1 9 Buses System

A smal-size system, composed by a hydroelectric unit, 3 thermal plants, 9 buses and 9 lines, is considered. A one-line diagram of the system is shown in Figure 1, where electrical elements are shown by standardized schematic symbols. The circles represent electric generators such as hydroelectric (H1) and thermal (T1, T2, T3) plants, while black bars represent the buses (N1-N9) and grey lines represent the interconnecting conductors. Other elements such as circuit breakers (red squares in Figure 1), inductors, transformers, capacitors, etc., are often indicated by specific symbols.

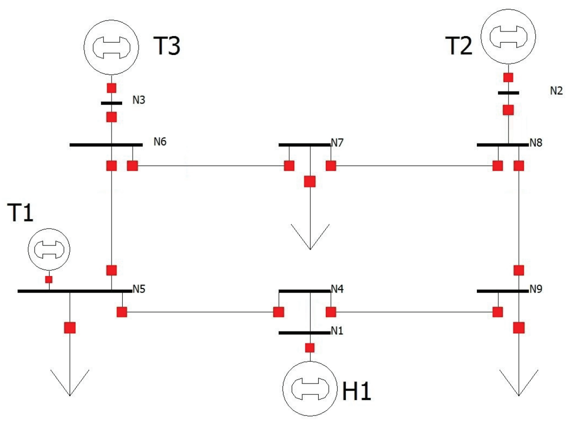

Figure 1: One-line diagram

Table 1: 9-buses lines capacity

	Line 1	Line 2	Line 3	Line 4	Line 5	Line 6	Line 7	Line 8	Line 9
Pattern A	150	250	150	150	200	150	150	150	250
Pattern B	150	250	150	100	90	100	150	150	250
Pattern C	100	250	100	100	90	70	100	150	150

As mentioned before, network constraints are governed by a simplified model (called DC because of its similarity with direct current flow equations); so system reactive power is not considered. Tables 1 and 2 show the specificity of the lines and the power plants for this model. The market comprises a hydroelectric company and two thermal companies, as shown in Table 2. Two different water availability scenarios were considered for the hydropower (scenarios 1 and 2), and three different capacity patterns were taken into account for the lines (patterns A, B and C). Thus six different numerical tests were performed (tests A1, A2, B1, B2, C1 and C2).

Table 2: 9-buses power units

Hydroelectric unit						
Company	Unit	Power		Efic. coeff.	Total Production	
		Min [MW]	Max [MW]	α	Scen.1	Scen.2
H 1	H_1	-50	240	1.05	320	640
Thermal units						
Company	Unit	Power		Costs		
		Min [MW]	Max [MW]	A $\left[\frac{U}{MWh^2}\right]$	B $\left[\frac{U}{MWh}\right]$	C [U]
Th 1	T_1	10	200	0.14	5	350
	T_2	10	220	0.15	5	350
Th 2	T_3	10	230	0.123	1	335

The system presents power demand on buses 5, 7 and 9. For calculating the anchor points (demand-price) only anchor demand values at these buses for each period were given. For this reason, prices for each demand level were calculated assuming that these demands were met considering only thermal generation. As a market price estimation, an aggregation of the marginal cost functions of several thermal plants is considered. The anchor point values for each period are defined in Table 3.

In order to estimate the price elasticity of demand (ε), the approach of [1] is followed. In particular, the market demand is estimated for $\varepsilon = -1/3$ measured at the anchor point. Accordingly, the slope parameter of the demand function is calculated considering that the elasticity at the demand level is equal to ε and the intercept is calculated so as to fit anchor quantity and anchor price at each demand level. The inverse demand function has the form $p = \frac{D}{a} - \frac{d}{a}$, where the obtained coefficient values are given in Table 3.

The numerical results are now presented. Tables 4, 5 and 6 show the obtained thermal and hydroelectric production patterns, as well as the final equilibrium market prices for the six numerical tests performed. The corresponding benefits and the algorithm performance (concerning number of iterations and computational time) are shown in Table 7.

As one would expect, large productions correspond to high prices periods and low productions correspond to low prices periods. Moreover, on water scarcity scenarios, it can be observed that the hydroelectric unit proceeds to pump water in low prices periods. This behavior disappears when water availability is sufficient and, in those situations, the corresponding hydroelectric benefits are much larger, while the thermal benefits decrease significantly. The equilibrium prices also decrese in high water avaibility periods, when thermal production is lower.

On the other hand, it is shown that modifications on the lines capacities can force changes in the production patterns, modifying the corresponding benefits. The experiments

Table 3: 9-buses demand-price anchor points and IDF coefficients

t	d_5	d_7	d_9	p	D_5	D_7	D_9	a_5	a_7	a_9
1	64.70	71.50	78.20	36.20	86.27	95.33	104.27	0.60	0.66	0.72
2	47.90	49.40	52.70	28.71	63.87	65.87	70.27	0.56	0.57	0.61
3	35.90	31.10	42.30	25.29	47.87	41.47	56.40	0.47	0.41	0.56
4	55.30	60.50	68.10	32.46	73.73	80.67	90.80	0.57	0.62	0.70
5	72.30	76.40	95.40	40.04	96.40	101.87	127.20	0.60	0.64	0.79
6	91.40	93.60	114.40	47.52	121.87	124.80	152.53	0.64	0.66	0.80
7	109.70	105.10	128.90	47.67	146.27	140.13	171.87	0.77	0.73	0.90
8	133.20	129.10	149.60	43.75	177.60	172.13	199.47	1.01	0.98	1.14
9	173.50	164.50	178.50	44.65	231.33	219.33	238.00	1.30	1.23	1.33
10	126.70	116.30	130.40	45.36	168.93	155.07	173.87	0.93	0.85	0.96
11	81.20	85.80	107.30	44.09	108.27	114.40	143.07	0.61	0.65	0.81
12	72.40	80.50	100.60	41.29	96.53	107.33	134.13	0.58	0.65	0.81

Table 4: 9-buses results. Production and prices A1-A2

	x_1	x_2	x_3	y	Price		x_1	x_2	x_3	y	Price
	44.72	40.21	92.82	13.32	48.02		40.20	36.97	85.07	36.59	44.09
T	32.70	28.74	68.41	0.00	40.28	T	28.97	26.06	62.00	19.23	36.60
E	24.37	21.18	51.31	-3.88	36.62	E	21.49	19.11	46.36	10.98	33.18
S	39.14	35.04	81.73	5.68	44.28	S	34.81	31.95	74.31	27.94	40.35
T	49.93	45.29	103.35	21.52	51.86	T	45.27	41.96	95.37	45.47	47.93
	59.87	54.25	122.52	37.94	59.34		55.06	50.81	114.27	62.69	55.41
A	69.08	61.88	140.57	43.78	59.49	A	63.59	57.92	131.12	72.11	55.56
1	83.97	74.73	171.25	44.86	55.57	2	76.80	69.57	158.92	81.85	51.64
	105.49	92.97	213.89	58.58	56.46		96.70	86.61	198.73	104.03	52.54
	76.19	66.99	154.15	43.64	57.18		69.94	62.46	143.37	75.98	53.25
	55.25	50.32	113.87	30.34	55.91		50.50	46.92	105.72	54.79	51.98
	51.38	47.07	106.64	24.22	53.11		46.69	43.73	98.60	48.34	49.18

also show that the equilibrium prices present a little increase in the cases when lines capacity patterns are tighter.

4.2 Neuquén-Rio Negro System

We now consider a section of the Argentinian National Interconnected System, whose transportation is managed by Transcomahue company. The system is located at Alto Valle and it comprises Neuquén and Rio Negro provinces, and so it is called Neuquén-Rio Negro system (NR). It is a medium size network with 23 thermal units, 6 hydroelectric plants, 87 buses and 89 lines. A one-line diagram of the system is shown in Figure 2.

Hydroelectric units are associated in 3 companies and thermal units in 6. In particular, hydroelectric plants belonging to the same company correspond to different generators on the same reservoir, and they are identical. As in the previous example, the network constraints are governed by a DC model and system reactive power is not considered.

It is important to mention that the real system has regular hydroelectric plants and not

Table 5: 9-buses results. Production and prices B1-B2

	x_1	x_2	x_3	y	Price		x_1	x_2	x_3	y	Price
	44.84	40.29	93.03	12.70	48.13		40.24	37.00	85.14	36.38	44.13
T	32.70	28.74	68.41	0.00	40.28	T	29.01	26.09	62.06	19.04	36.63
E	24.45	21.24	51.46	-4.31	36.72	E	21.52	19.13	46.41	10.82	33.21
S	39.25	35.12	81.93	5.09	44.39	S	34.85	31.97	74.38	27.74	40.39
T	50.06	45.38	103.56	20.87	51.97	T	45.32	41.99	95.44	45.25	47.97
	60.00	54.34	122.74	37.28	59.45		55.10	50.84	114.34	62.47	55.45
B	69.22	61.98	140.82	43.02	59.60	B	63.64	57.96	131.21	71.85	55.60
1	82.43	76.60	171.58	43.87	55.67	2	74.85	71.63	159.03	81.51	51.68
	99.88	107.09	190.00	65.46	58.67		64.82	121.59	190.00	106.57	53.34
	76.35	67.11	154.44	42.77	57.28		69.99	62.50	143.47	75.69	53.28
	55.38	50.41	114.09	29.69	56.01		50.54	46.96	105.80	54.57	52.01
	51.51	47.16	106.86	23.57	53.21		46.73	43.76	98.67	48.12	49.22

Table 6: 9-buses results. Production and prices C1-C2

	x_1	x_2	x_3	y	Price		x_1	x_2	x_3	y	Price
	45.05	40.44	93.39	11.61	48.31		40.13	36.92	84.95	36.93	44.04
T	32.73	28.77	68.47	-0.17	40.31	T	28.91	26.02	61.90	19.53	36.54
E	24.60	21.35	51.71	-5.07	36.89	E	21.44	19.07	46.28	11.22	33.12
S	39.45	35.27	82.27	4.05	44.57	S	34.75	31.90	74.20	28.26	40.30
T	50.27	45.53	103.94	19.75	52.15	T	45.21	41.91	95.25	45.81	47.88
	57.87	56.85	123.13	36.12	59.63		52.51	53.24	114.15	63.05	55.35
C	64.97	66.68	141.26	41.69	59.78	C	58.85	62.52	130.99	72.52	55.50
1	78.91	84.75	160.00	46.19	57.15	2	54.01	92.18	158.74	82.39	51.58
	86.57	131.34	160.00	73.57	61.51		55.73	148.97	160.00	100.00	58.08
	70.72	73.25	154.95	41.26	57.47		64.06	68.17	143.22	76.45	53.19
	54.54	51.64	114.47	28.55	56.19		49.37	47.94	105.60	55.15	51.92
	51.55	47.49	107.23	22.45	53.40		46.42	43.88	98.48	48.69	49.12

pumped-storage units, so the used data were taken from literature examples. The goal of introducing pumping capacity into the model is to study the existence of scenarios where this capability is beneficial, in order to assess the benefits that such units can bring. Various numerical experiments were performed, with three different scenarios of water availability in the planning horizon, and allowing or not pumping.

Similarly to the 9-buses example, the anchor prices were calculated from buses anchor demands as an aggregation of the marginal cost functions of the thermal plants which satisfy the anchor demand. The IDF coefficients on these buses were determined from demand-price anchor points for each period and the price elasticity of demand equal to $-1/3$.

Since data related to this example is too extensive, detailed information is not given here. Complete information including the characteristics of the thermal and hydroelectric units, the lines capacity and the description of the water availability scenarios, as well as the anchor points for the expected total demand and the prices with the corresponding obtained IDF coefficients, can be found in the website `http://nas1.pladema.net/shares`

Table 7: 9-buses results. Benefits and algoritmh performance

Test	A1	A2	B1	B2	C1	C2
Ben_1^{Th}	49463.85	40555.07	50429.77	40733.57	52096.15	41862.45
Ben_2^{Th}	58938.75	49932.24	58831.60	49912.39	58352.46	49568.62
Ben^H	18290.51	32431.75	18517.80	32547.29	18960.18	32938.35
Iterations	476	477	470	591	581	552
Comp. time (s)	0.36	0.39	0.36	0.47	0.45	0.42

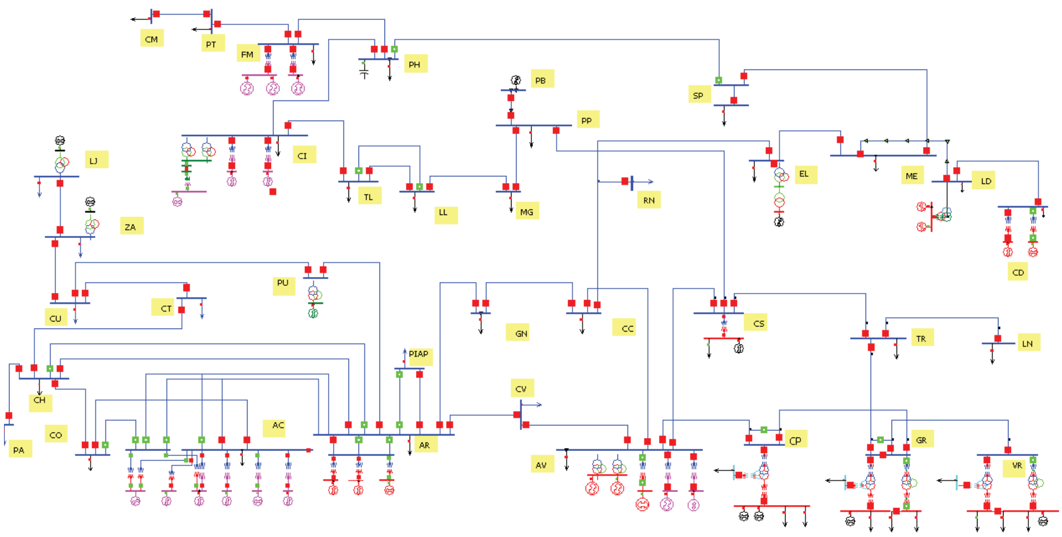

Figure 2: NR one-line diagram

/Publico/Neuquen-RN-data.pdf .

Figure 3 shows the average production/consumption for hydroelectric companies and the equilibrium prices for each period, for those scenarios in which pumping is allowed. Table 8 shows the obtained benefits and the average price of the analysis horizon in different scenarios, as well as the algorithm performance concerning number of iterations and computational time.

As might be expected, in low water availability scenarios, hydroelectric companies pump water on low demand-prices periods, in order to generate energy in the most profitable periods. Pumping decreases or disappears as water availability is greater. In particular, in scenario 3 there is no need for pumping, so the results for when pumping is allowed are the same as for the case in which this possibility is restricted. It can be observed that water scarcity tends to increase the electricity price, according to the intensive use of the more expensive thermal units.

On the other hand, hydroelectric companies that opt to pump when permitted, get significantly lower benefits if the possibility of pumping is restricted. This prohibition also results in higher profits for thermal companies. And it is quite remarkable that the use of pumping allows to decrese the average electricity price.

Finally, concerning the algorithm's performance, it can be seen that despite of the large

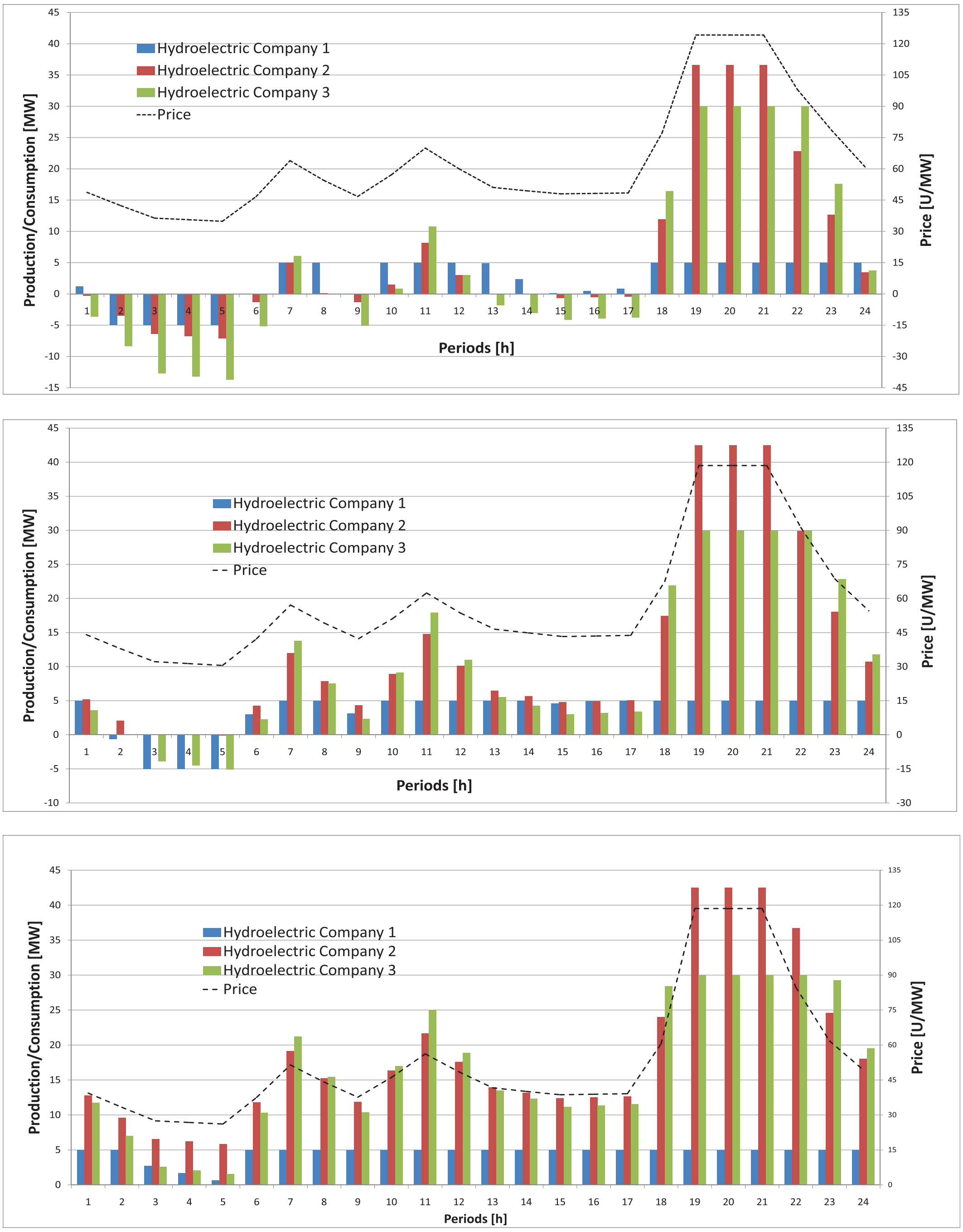

Figure 3: NR results. Price and hydroelectric activity with pumping in scenarios 1 , 2 and 3

Table 8: NR results. Benefits, average prices and performance

Scenario	1	2	3	1	2	3
Pumping	Yes	Yes	Yes	No	No	No
BenH1	6240.60	6641.37	6157.52	4756.57	5468.66	6157.52
BenH2	61035.40	77130.29	94010.65	52809.51	77256.53	94010.65
BenH3	43501.56	44177.14	53167.44	25261.55	42012.56	53167.44
BenTh1	101243.84	84575.01	71628.11	107695.53	85046.08	71628.13
BenTh2	152058.45	131559.69	118192.72	164745.08	131733.45	118192.72
BenTh3	76899.86	61030.22	49593.20	82826.84	61478.16	49593.20
BenTh4	43564.13	36610.80	30335.61	45322.06	36674.02	30335.61
BenTh5	46777.93	36732.43	28640.16	49179.72	36991.66	28640.16
BenTh6	24320.11	13204.14	5276.88	27992.78	13581.48	5276.88
Av. price	63,72	58,06	53,53	65,51	58,12	53,53
Iterations	6788	6650	6696	6807	6273	6696
Time (s)	144	140	142	143	135	143

number of iterations, the computational cost of each of them is very low and so the total computational times are very reasonable.

5 Conclusions

In previous works, the numerical resolution of models for the behavior of electricity generating companies acting in an oligopolistic market was considered. To gain more realism, in this work a new kind of constraints associated to the network characteristics is included. The methodology previously applied to the simpler model is extended to solve the new more realistic one. The variational inclusion derived from the model is tackled with a variable metric proximal decomposition method. The inclusion represents mathematically the coupled-in-time Nash-Cournot equilibrium for the scheduling of hydroelectricity production considering all the market players and also the network constraints.

The proposed methodology was effective for small-size and medium-size networks. In the studied examples, the numerical experiments verify that the possibility of pumping water back can deliver better profits in water-stressed scenarios. When water is available, pumping is no longer needed. Also, it is shown that modifications in the lines capacities can force changes in the production patterns, altering the corresponding benefits.

Numerical results obtained for a medium-size real case show that this methodology could be applied both for the scheduling of a heterogeneous network of power generation units and for the analysis of the market and the policies of the regulatory agencies.

6 APPENDIX: The Decomposition Scheme

Our terminology follows [14]. In short, for the application in consideration we need to show that NCNCEP can be formulated as a variational inclusion with the following structure:

$$0 \in T(u,v) = F(u,v) \times [G(u,v) + H(v)], \tag{6.1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ are such that

A1 G is a (single-valued) continuous function.

A2 H is maximal monotone.

A3 The mapping $(u,v) \mapsto F(u,v) \times G(u,v)$ is maximal monotone.

A4 $\operatorname{dom} H \subset \operatorname{rint}\{v \in \mathbb{R}^m \mid \exists u \in \mathbb{R}^n \text{ s.t. } F(u,v) \times G(u,v) \neq \emptyset\}$.

Under these assumptions, it follows that T is maximal monotone [13], and that the mapping $u \to F(u,v)$ is maximal monotone for any fixed $v \in \operatorname{dom} H$ [20, Lemma 2.1]. VMHPDM is derived from the Variable Metric Hybrid Inexact Proximal Point Method presented in [9], and both schemes are extensions to the variable metric setting of the methods introduced in [18] and [19], respectively.

We next explain the (simplified version of the) method. Given $(u^k, v^k) \in \mathbb{R}^n \times \mathbb{R}^m$, first a *forward-backward splitting* step is performed with the u-part fixed, computing

$$\hat{v}^k = (I + c_k Q_k H(\cdot))^{-1} \left(I - c_k Q_k G(u^k, v^k)\right), \tag{6.2}$$

where $c_k > 0$ and Q_k is a symmetric positive definite matrix. This step splits the sum $G+H$ "forward" in G and "backward" in H.

The splitting step is followed by an inexact proximal step with the v-part fixed, which means computing an approximation of

$$\hat{u}^k = \left(c_k P_k F(\cdot, \hat{v}^k) + I\right)^{-1} (u^k), \tag{6.3}$$

where P_k is a symmetric positive definite matrix. By inexactness we mean that instead of computing $\hat{u}^k$ in (6.3), any $\hat{u}^k$ satisfying the proximal approximation condition

$$\|s^k\|^2_{Q_k^{-1}} \leq \sigma_k^2 \left(\|\hat{u}^k - u^k\|^2_{P_k^{-1}} + \|\hat{v}^k - v^k\|^2_{Q_k^{-1}} \right), \tag{6.4}$$

is acceptable, where $\sigma_k \in (0,1)$ and $s^k = c_k Q_k \left(G(\hat{u}^k, \hat{v}^k) - G(u^k, v^k)\right)$. In its general form, the method uses an even weaker approximation condition, which can be always satisfied provided the splitting step has been solved with sufficient accuracy and with c_k small enough; see [7]. For our application, we have proven that the stronger condition (6.4) can be satisfied for a sufficiently small c_k. We also note that inverting the matrices is not required for checking (6.4); see [9].

Having performed the splitting and proximal steps, the next iterates are obtained by setting

$$\begin{aligned} u^{k+1} &= \hat{u}^k, \\ v^{k+1} &= \hat{v}^k - s^k. \end{aligned} \tag{6.5}$$

The decomposition framework outlined above is rather general; for example, it contains as special cases the methods in [3, 6, 11, 18, 20]. The described algorithm converges globally to a solution of (6.1), with linear local rate of convergence under suitable assumptions [7, Theorem 2]. For a full description and further discussion, see [7].

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